Supplemental problems: Chapter 6

1. True or false. If the statement is always true, answer true. Otherwise, answer false. Justify your answer.

a) Suppose $W = \text{Span}\{w\}$ for some vector $w \neq 0$, and suppose v is a vector orthogonal to w. Then the orthogonal projection of v onto W is the zero vector.

b) Suppose W is a subspace of $\mathbf{R}^n$ and x is a vector in $\mathbf{R}^n$. If x is not in W, then $x - x_W$ is not zero.

c) Suppose W is a subspace of $\mathbf{R}^n$ and x is in both W and $W^\perp$. Then $x = 0$.

d) Suppose $\widehat{x}$ is a least squares solution to $Ax = b$. Then $\widehat{x}$ is the closest vector to b in the column space of A.

Solution.

a) True. Since $v \in W^\perp$, its projection onto W is zero.

b) True. If x is not in W then $x \neq x_W$, so $x - x_W$ is not zero.

c) True. Since x is in W and $W^\perp$ it is orthogonal to itself, so $||x||^2 = x \cdot x = 0$. The length of x is zero, which means every entry of x is zero, hence $x = 0$.

d) False: $A\widehat{x}$ is the closest vector to b in Col A.

2. Let $W = \text{Span}\{v_1, v_2\}$, where $v_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

a) Find the closest point w in W to $x = \begin{pmatrix} 0 \\ 14 \\ -4 \end{pmatrix}$.

Let $A = \begin{pmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{pmatrix}$. We solve $A^T A v = A^T x$.

$$A^T A = \begin{pmatrix} 6 & 6 \\ 6 & 14 \end{pmatrix} \qquad A^T \begin{pmatrix} 0 \\ 14 \\ -4 \end{pmatrix} = \begin{pmatrix} 24 \\ 16 \end{pmatrix}.$$

We find $\left(\begin{array}{cc|c} 6 & 6 & 24 \\ 6 & 14 & 16 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -1 \end{array}\right)$, so $v = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$ and therefore

$$w = Av = \begin{pmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -6 \\ 8 \\ 2 \end{pmatrix}.$$

b) Find the distance from w to $\begin{pmatrix} 0 \\ 14 \\ -4 \end{pmatrix}$.

$$||x-w|| = \left\| \begin{pmatrix} 0 \\ 14 \\ -4 \end{pmatrix} - \begin{pmatrix} -6 \\ 8 \\ 2 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 6 \\ 6 \\ -6 \end{pmatrix} \right\| = \sqrt{36+36+36} = \sqrt{108} = 6\sqrt{3}.$$

c) Find the standard matrix for the orthogonal projection onto Span$\{v_1\}$.

$$B = \frac{1}{v_1 \cdot v_1} v_1 v_1^T = \frac{1}{(-1)^2+2^2+1^2} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} -1 & 2 & 1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 & -2 & -1 \\ -2 & 4 & 2 \\ -1 & 2 & 1 \end{pmatrix}$$

d) Find the standard matrix for the orthogonal projection onto W.

Let $A = \begin{pmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{pmatrix}$. Since the columns of A are linearly independent, our projection matrix is $A(A^TA)^{-1}A^T$. We already computed A^TA in part (a), so our matrix is

$$A(A^TA)^{-1}A^T = \begin{pmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 6 & 6 \\ 6 & 14 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 6 & 6 \\ 6 & 14 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

$$= \frac{1}{48} \begin{pmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 14 & -6 \\ -6 & 6 \end{pmatrix} \begin{pmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

3. Find the least-squares line $y = Mx + B$ that approximates the data points

$$(-2,-11), \quad (0,-2), \quad (4,2).$$

Solution.

If there were a line through the three data points, we would have:

$$(x=-2) \qquad B + M(-2) = -11$$
$$(x=0) \qquad B + M(0) = -2$$
$$(x=4) \qquad B + M(4) = 2.$$

This is the matrix equation $\begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} B \\ M \end{pmatrix} = \begin{pmatrix} -11 \\ -2 \\ 2 \end{pmatrix}$.

Thus, we are solving the least-squares problem to $Av = b$, where

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 4 \end{pmatrix} \qquad b = \begin{pmatrix} -11 \\ -2 \\ 2 \end{pmatrix}.$$

We solve $A^T A\widehat{x} = A^T b$, where $\widehat{x} = \begin{pmatrix} B \\ M \end{pmatrix}$.

$$A^T A = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 20 \end{pmatrix},$$

$$A^T b = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 4 \end{pmatrix} \begin{pmatrix} -11 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ 30 \end{pmatrix}.$$

$$\left(\begin{array}{cc|c} 3 & 2 & -11 \\ 2 & 20 & 30 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 2 & 20 & 30 \\ 3 & 2 & -11 \end{array}\right) \xrightarrow[R_1 = R_1/2]{R_2 = R_2 - \frac{3R_1}{2}} \left(\begin{array}{cc|c} 1 & 10 & 15 \\ 0 & -28 & -56 \end{array}\right) \xrightarrow[R_1 = R_1 - 10R_2]{R_2 = -\frac{R_2}{28}} \left(\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & 2 \end{array}\right).$$

So $\widehat{x} = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$. In other words, $y = -5 + 2x$, or $\boxed{y = 2x - 5}$.