Recall: an $n \times n$ matrix A is **diagonalizable** if it is similar to a diagonal matrix:

$$A = CDC^{-1} \quad \text{for} \quad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

It is easy to take powers of diagonalizable matrices:

$$A^i = CD^iC^{-1} = C \begin{pmatrix} \lambda_1^i & 0 & \cdots & 0 \\ 0 & \lambda_2^i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^i \end{pmatrix} C^{-1}.$$
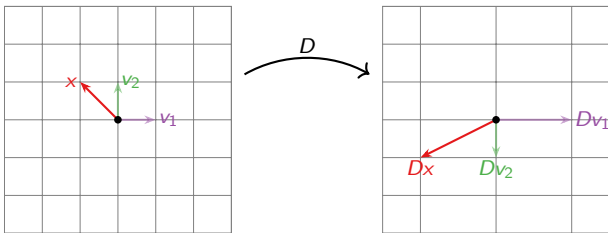
We begin today by discussing the *geometry* of diagonalizable matrices.

A diagonal matrix $D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ just scales the coordinate axes:

$$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This is easy to visualize:

$$x = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \Longrightarrow \qquad Dx = \begin{pmatrix} -2 \\ -1 \end{pmatrix}.$$

We had this example last time: $A = CDC^{-1}$ for

$$A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$
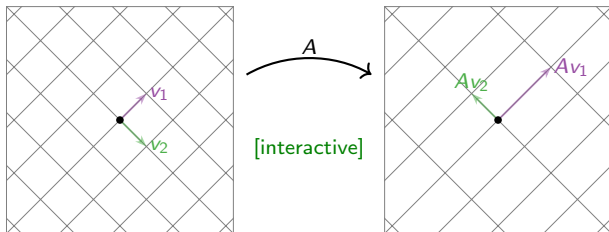
The eigenvectors of A are $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with eigenvalues 2 and -1.

The eigenvectors form a *basis* for $\mathbf{R}^2$ because they're linearly independent.

Any vector can be written as a linear combination of basis vectors:

$$x = a_1 v_1 + a_2 v_2 \implies Ax = A(a_1 v_1 + a_2 v_2) = a_1 A v_1 + a_2 A v_2 = 2a_1 v_1 - a_2 v_2.$$

Conclusion: A scales the "v_1-direction" by 2 and the "v_2-direction" by -1.
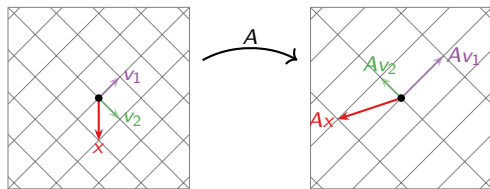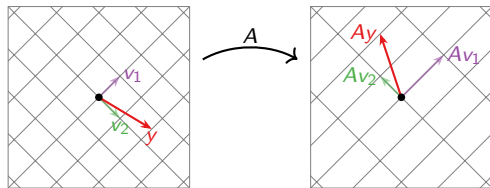
Example: $x = \begin{pmatrix} 0 \\ -2 \end{pmatrix} = -1v_1 + 1v_2$

$$Ax = -1Av_1 + 1Av_2 = -2v_1 + -1v_2 = -2\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}.$$

Example: $y = \frac{1}{2}\begin{pmatrix} 5 \\ -3 \end{pmatrix} = \frac{1}{2}v_1 + 2v_2$

$$Ay = \frac{1}{2}Av_1 + 2Av_2 = 1v_1 + -2v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}.$$

We motivated diagonalization by taking powers:

$$A^i = CD^iC^{-1} = C \begin{pmatrix} \lambda_1^i & 0 & \cdots & 0 \\ 0 & \lambda_2^i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^i \end{pmatrix} C^{-1}.$$

This lets us compute powers of matrices easily. How to visualize this?

$$A^n v = A(A(A \cdots (Av)) \cdots)$$

Multiplying a vector v by A^n means repeatedly multiplying by A.

$$A = \frac{1}{10} \begin{pmatrix} 11 & 6 \\ 9 & 14 \end{pmatrix} = CDC^{-1} \quad \text{for} \quad C = \begin{pmatrix} 2/3 & 1 \\ -1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Eigenvectors of A are $v_1 = \binom{2/3}{-1}$ and $v_2 = \binom{1}{1}$ with eigenvalues 2 and 1/2.

$$A(a_1 v_1 + a_2 v_2) = 2 a_1 v_1 + \frac{1}{2} a_2 v_2$$

$$A^2(a_1 v_1 + a_2 v_2) = 4 a_1 v_1 + \frac{1}{4} a_2 v_2$$

$$A^3(a_1 v_1 + a_2 v_2) = 8 a_1 v_1 + \frac{1}{8} a_2 v_2$$

$$\vdots$$

$$A^n(a_1 v_1 + a_2 v_2) = 2^n a_1 v_1 + \frac{1}{2^n} a_2 v_2$$

What does repeated application of A do geometrically?

It makes the "v_1-coordinate" very big, and the "v_2-coordinate" very small.

[interactive]

Dynamics of Diagonalizable Matrices
Another Example

$$A = \frac{1}{6}\begin{pmatrix} 5 & -1 \\ -2 & 4 \end{pmatrix} = CDC^{-1} \quad \text{for} \quad C = \begin{pmatrix} -1 & 1/2 \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Eigenvectors of A are $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$ with eigenvalues 1 and 1/2.

$$A(a_1 v_1 + a_2 v_2) = a_1 v_1 + \frac{1}{2} a_2 v_2$$

$$A^2(a_1 v_1 + a_2 v_2) = a_1 v_1 + \frac{1}{4} a_2 v_2$$

$$A^3(a_1 v_1 + a_2 v_2) = a_1 v_1 + \frac{1}{8} a_2 v_2$$

$$\vdots$$

$$A^n(a_1 v_1 + a_2 v_2) = a_1 v_1 + \frac{1}{2^n} a_2 v_2$$

What does repeated application of A do geometrically?

It "sucks everything into the 1-eigenspace."

[interactive]

$$A = \frac{1}{30} \begin{pmatrix} 12 & 2 \\ 3 & 13 \end{pmatrix} = CDC^{-1} \quad \text{for} \quad C = \begin{pmatrix} 2/3 & -1 \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix}.$$

Poll

What does repeated application of A do geometrically?

   A. Sucks all vectors into a line.

   B. Sucks all vectors into the origin.

   C. Shoots all vectors away from a line.

   D. Shoots all vectors away from the origin.

B. Since both eigenvalues are less than 1, the matrix A scales *both* directions towards the origin.
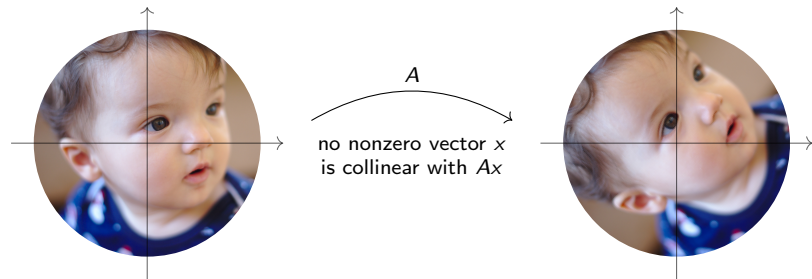
[interactive]

Section 5.5

Complex Eigenvalues

A Matrix with No Eigenvectors

Consider the matrix for the linear transformation for rotation by $\pi/4$ in the plane. The matrix is:

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

This matrix has no eigenvectors, as you can see geometrically: [interactive]

A

no nonzero vector x
is collinear with Ax

or algebraically:

$$f(\lambda) = \lambda^2 - \operatorname{Tr}(A)\,\lambda + \det(A) = \lambda^2 - \sqrt{2}\,\lambda + 1 \implies \lambda = \frac{\sqrt{2} \pm \sqrt{-2}}{2}.$$

It makes us sad that -1 has no square root. If it did, then $\sqrt{-2} = \sqrt{2} \cdot \sqrt{-1}$.

Mathematician's solution: we're just not using enough numbers! We're going to declare by *fiat* that there exists a square root of -1.

Definition
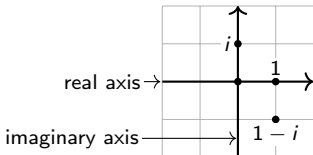The number i is defined such that $i^2 = -1$.

Once we have i, we have to allow numbers like $a + bi$ for real numbers a, b.

Definition
A *complex number* is a number of the form $a + bi$ for a, b in $\mathbf{R}$. The set of all complex numbers is denoted $\mathbf{C}$.

Note $\mathbf{R}$ is contained in $\mathbf{C}$: they're the numbers $a + 0i$.

We can identify $\mathbf{C}$ with $\mathbf{R}^2$ by $a + bi \longleftrightarrow \binom{a}{b}$. So when we draw a picture of $\mathbf{C}$, we draw the plane:

Operations on Complex Numbers

Addition: $(2 - 3i) + (-1 + i) = 1 - 2i$.

Multiplication: $(2 - 3i)(-1 + i) = 2(-1) + 2i + 3i - 3i^2 = -2 + 5i + 3 = 1 + 5i$.

Complex conjugation: $\overline{a + bi} = a - bi$ is the **complex conjugate** of $a + bi$.
Check: $\overline{z + w} = \bar{z} + \bar{w}$ and $\overline{zw} = \bar{z} \cdot \bar{w}$.

Absolute value: $|a + bi| = \sqrt{a^2 + b^2}$. This is a *real* number.
Note: $(a + bi)(\overline{a + bi}) = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2$. So $|z| = \sqrt{z\bar{z}}$.
Check: $|zw| = |z| \cdot |w|$.

Division by a nonzero real number: $\dfrac{a + bi}{c} = \dfrac{a}{c} + \dfrac{b}{c}i$.

Division by a nonzero complex number: $\dfrac{z}{w} = \dfrac{z\bar{w}}{w\bar{w}} = \dfrac{z\bar{w}}{|w|^2}$.

Example:
$$\frac{1 + i}{1 - i} = \frac{(1 + i)^2}{1^2 + (-1)^2} = \frac{1 + 2i + i^2}{2} = i.$$

Real and imaginary part: $\operatorname{Re}(a + bi) = a \qquad \operatorname{Im}(a + bi) = b$.

The whole point of using complex numbers is to solve polynomial equations. It turns out that they are enough to find all solutions of all polynomial equations:

Fundamental Theorem of Algebra

Every polynomial of degree n has exactly n complex roots, counted with multiplicity.

Equivalently, if $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree n, then

$$f(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)$$

for (not necessarily distinct) complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Important

If f is a polynomial with *real* coefficients, and if λ is a complex root of f, then so is $\overline{\lambda}$:

$$
\begin{aligned}
0 = \overline{f(\lambda)} &= \overline{\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0} \\
&= \overline{\lambda}^n + a_{n-1}\overline{\lambda}^{n-1} + \cdots + a_1\overline{\lambda} + a_0 = f\left(\overline{\lambda}\right).
\end{aligned}
$$

Therefore complex roots of real polynomials come in *conjugate pairs*.

Degree 2: The quadratic formula gives you the (real or complex) roots of any degree-2 polynomial:

$$f(x) = x^2 + bx + c \implies x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

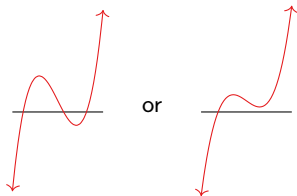For instance, if $f(\lambda) = \lambda^2 - \sqrt{2}\lambda + 1$ then

$$\lambda = \frac{\sqrt{2} \pm \sqrt{-2}}{2} = \frac{\sqrt{2}}{2}(1 \pm i) = \frac{1 \pm i}{\sqrt{2}}.$$

Note the roots are complex conjugates if b, c are real.

Degree 3: A real cubic polynomial has either three real roots, or one real root and a conjugate pair of complex roots. The graph looks like:

or

respectively.

A Matrix *with* an Eigenvector

Every matrix is guaranteed to have *complex* eigenvalues and eigenvectors.
Using rotation by $\pi/4$ from before:

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{has eigenvalues} \quad \lambda = \frac{1 \pm i}{\sqrt{2}}.$$

Let's compute an eigenvector for $\lambda = (1 + i)/\sqrt{2}$:

$$A - \lambda I = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 - (1+i) & -1 \\ 1 & 1 - (1+i) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}.$$

The second row is i times the first, so we row reduce:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \rightsquigarrow \frac{1}{\sqrt{2}} \begin{pmatrix} -i & -1 \\ 0 & 0 \end{pmatrix} \xrightarrow{\text{divide by } -i/\sqrt{2}} \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}.$$

The parametric form is $x = iy$, so an eigenvector is $\begin{pmatrix} i \\ 1 \end{pmatrix}$. So is *any* nonzero

complex scalar multiple of $\begin{pmatrix} i \\ 1 \end{pmatrix}$, for example $(-i/\sqrt{2}) \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$.

A similar computation shows that an eigenvector for $\lambda = (1 - i)/\sqrt{2}$ is $\begin{pmatrix} -i \\ 1 \end{pmatrix}$.

So is $i \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ (you can scale by *complex* numbers).

For $A = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$,

the eigenvalue $\dfrac{1+i}{\sqrt{2}}$ has eigenvector $\begin{pmatrix} i \\ 1 \end{pmatrix}$.

the eigenvalue $\dfrac{1-i}{\sqrt{2}}$ has eigenvector $\begin{pmatrix} -i \\ 1 \end{pmatrix}$.

Do you notice a pattern?

Fact

Let A be a real square matrix. If λ is a complex eigenvalue with eigenvector v, then $\bar{\lambda}$ is an eigenvalue with eigenvector $\bar{v}$.

Why?

$$Av = \lambda \implies A\bar{v} = \overline{Av} = \overline{\lambda v} = \bar{\lambda}\bar{v}.$$

Both eigenvalues and eigenvectors of real square matrices occur in conjugate pairs.

Suppose A is a 2×2 matrix and λ is any eigenvalue of A. Then

$$A - \lambda I_2 = \begin{pmatrix} z & w \\ (*) & (*) \end{pmatrix} \implies \begin{pmatrix} -w \\ z \end{pmatrix}$$

is an eigenvector of A corresponding to the eigenvalue λ.

In the previous example, $\lambda = \dfrac{1+i}{\sqrt{2}}$ was an eigenvalue of $A = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$
and

$$A - \lambda I = \begin{pmatrix} -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} z & w \\ (*) & (*) \end{pmatrix}.$$

So an eigenvector of A corresponding to λ is

$$v = \begin{pmatrix} -w \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}.$$

This was much faster than doing the full 2×2 row reduction in the previous example, and it agrees with our answer.

Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The characteristic polynomial is

$$f(\lambda) = \det \begin{pmatrix} \frac{4}{5} - \lambda & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} - \lambda & 0 \\ 0 & 0 & 2 - \lambda \end{pmatrix} = \boxed{(2 - \lambda)} \left(\lambda^2 - \frac{8}{5}\lambda + 1 \right).$$

This factors out automatically if you expand cofactors along the third row or column

We computed the roots of this polynomial (times 5) before:

$$\lambda = \quad 2, \quad \frac{4 + 3i}{5}, \quad \frac{4 - 3i}{5}.$$

We eyeball an eigenvector with eigenvalue 2 as $(0, 0, 1)$.

$$A = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

To find the other eigenvectors, we row reduce:

$$A - \frac{4+3i}{5}I = \begin{pmatrix} -\frac{3}{5}i & -\frac{3}{5} & 0 \\ \frac{3}{5} & -\frac{3}{5}i & 0 \\ 0 & 0 & 2 - \frac{4+3i}{5} \end{pmatrix} \xrightarrow{\text{scale rows}} \begin{pmatrix} -i & -1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The second row is i times the first:

$$\xrightarrow{\text{row replacement}} \begin{pmatrix} -i & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{divide by } -i,\text{swap}} \begin{pmatrix} 1 & -i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The parametric form is $x = iy$, $z = 0$, so an eigenvector is $\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$. Therefore, an

eigenvector with conjugate eigenvalue $\dfrac{4-3i}{5}$ is $\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}$.

Summary

- Diagonal matrices are easy to understand geometrically.
- Diagonalizable matrices behave like diagonal matrices, except with respect to a basis of eigenvectors.
- Repeatedly multiplying by a matrix leads to fun pictures.
- One can do arithmetic with complex numbers just like real numbers: add, subtract, multiply, divide.
- An $n \times n$ matrix always exactly has complex n eigenvalues, counted with (algebraic) multiplicity.
- The complex eigenvalues and eigenvectors of a *real* matrix come in complex conjugate pairs:

$$Av = \lambda v \quad \implies \quad A\bar{v} = \bar{\lambda}\bar{v}.$$