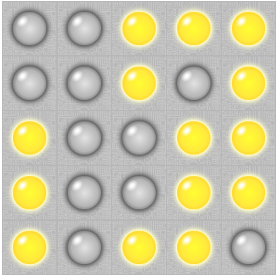
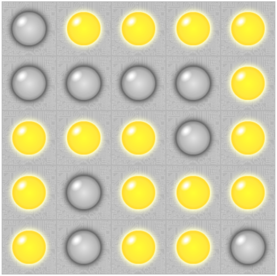
Lights Out

An application of linear algebra to something important

Lights Out

When you click on a light, it toggles that light and the 4 lights immediately adjacent. If the light is on the edge, then there are only 3 adjacent lights. If the light is on the corner, there are only 2 adjacent lights. The goal is to turn all the lights off.





If you click on the center square, it changes the first configuration to the second (and vice versa!).

Play

Modular arithmetic

aka Clock arithmetic

You are used to clock arithmetic: $10 + 4 = 2$, $7 + 6 = 1$, $7 - 6 = 1$, etc.

Clock arithmetic is called mod 12 arithmetic. In mod 5 arithmetic, the numbers are $0, 1, 2, 3, 4$, and we can write $4 + 2 = 1$, $3 - 4 = 4$, $3 + 2 = 0$, etc.

For Lights Out, you use the simplest version, mod 2 arithmetic. There are two numbers, 0 and 1, and

$$0 + 0 = 0$$
$$0 + 1 = 1$$
$$1 + 0 = 1$$
$$1 + 1 = 0$$

Those are all the ways of adding two numbers in mod 2 arithmetic. Also, you don't need minus signs since $-0 = 0$ and $-1 = 1$.

Mod 2 linear algebra

It turns out that all of linear algebra works with mod 2 arithmetic. Consider the following system of equations:

$$\begin{aligned} x + y \phantom{{}+z} &= 1 \\ y + z &= 1 \\ x + y + z &= 0 \end{aligned}$$

Let's make a matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right)$$

Try row reducing and solving!

Mod 2 linear algebra

Here is the row reduction

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

There is no pivot in the last column, so the system is consistent. There is a pivot in each column, so there is a unique solution:

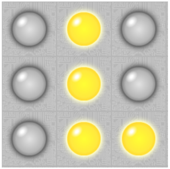
$$x = 1, \; y = 0, \; z = 1.$$

Lights out

Target vectors

What does this have to do with lights out? In the $n \times n$ version, you can think of a configuration of lights as a vector that has n^2 entries, and with each entry being a 0 or a 1. This is the target vector.

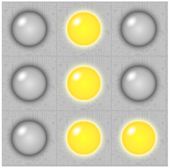
What is the target vector for this 3×3 game?



Lights out

Target vectors

What is the target vector for this 3×3 game?



Answer:

$$b = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \text{top left} \\ \text{top middle} \\ \text{top right} \\ \text{middle left} \\ \text{middle middle} \\ \text{middle right} \\ \text{bottom left} \\ \text{bottom middle} \\ \text{bottom right} \end{pmatrix}$$

Lights out

Target vectors

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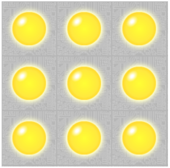

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Target vectors

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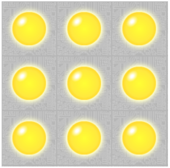


Answer:

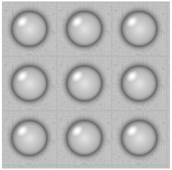
$$b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

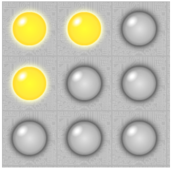
Lights out

Toggle vectors

Each time you press a button, you are adding a vector to the starting vector. The vector you are adding has 3, 4, or 5 entries equal to 1, and the other entries are equal to 0 (the ones are for the lights getting toggled). There are n^2 such toggle vectors in the $n \times n$ game, since you can click in n^2 places. Each vector has n^2 entries.

What is the toggle vector for clicking on the top left corner?





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Toggle vectors

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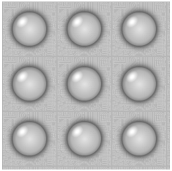
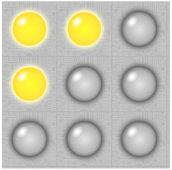




Answer:

$$c_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

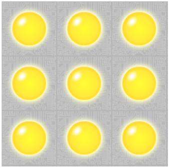
Lights out

When you click on a square in the game, you are adding a toggle vector c_i to the target vector b. You can add as many c_i as you want. The goal is to get 0.

So the whole game boils down to:

> *Find a linear combination of the toggle vectors c_i that equals the target vector b.*

See if you can solve this lights out game using linear algebra.



You are solving $Ax = b$ where the columns of A are the toggle vectors (so it is a 9×9 matrix) and b is the target vector $(1, 1, 1, 1, 1, 1, 1, 1, 1)$ given above.

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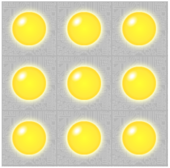


You are solving $Ax = b$ where the columns of A are the toggle vectors (so it is a 9×9 matrix) and b is the target vector $(1, 1, 1, 1, 1, 1, 1, 1, 1)$ given above.

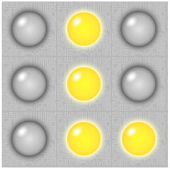
The solution is:

$$x = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

In other words, to turn all the lights off, you should click the four corners and the middle.

Lights out

Here is another one to try.



Challenge questions

Question. Is every lights out game solvable? If there a configuration of lights that cannot be turned off? What is the answer for the 3×3 game? 5×5? What about $n \times n$?

Question. Write computer code to solve any Lights Out game.

Question. Come up with an alternate version of the game. (Code it!) Is your version of the game always solvable?

Come up with your own questions!