

Math 1553 Worksheet §2.5, 2.6, 2.7, 2.9, 3.1

Solutions

1. If the statement is always true, circle TRUE. Otherwise, circle FALSE. Justify your answer.

a) Suppose  $A = (v_1 \ v_2 \ v_3)$  and  $A \begin{pmatrix} -3 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ . Must  $v_1, v_2, v_3$  be linearly dependent? If yes, write a linear dependence relation for the vectors.

YES      NO

b) If  $b$  is in  $\text{Col}(A)$ , then so is  $5b$ .      TRUE      FALSE

c) In the following,  $A$  is an  $m \times n$  matrix.

(1) TRUE      FALSE      If  $A$  has linearly dependent columns, then  $m < n$ .

(2) TRUE      FALSE      If  $A$  has linearly independent columns, then  $Ax = b$  must have at least one solution for each  $b$  in  $\mathbf{R}^m$ .

(3) TRUE      FALSE      If  $b$  is a vector in  $\mathbf{R}^m$  and  $Ax = b$  has exactly one solution, then  $m \geq n$ .

**Solution.**

a) **YES.** By definition of matrix multiplication,  $-3v_1 + 2v_2 + 7v_3 = 0$ , so  $\{v_1, v_2, v_3\}$  is linearly dependent and the equation gives a linear dependence relation.

b) **TRUE.** Let  $v$  be a solution to  $Ax = b$ , so  $Av = b$ . Then  $A(5v) = 5Av = 5b$ .

c) (1) **FALSE** For example  $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ .

(Note that even though this part was false, there is a very similar-sounding statement that is true: if  $m < n$ ,  $A$  must have linearly dependent columns.)

(2) **FALSE** For example  $A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . There is no solution for  $Ax = b$ .

(Note, however: if  $A$  has linearly independent columns, then the system  $Ax = 0$  has no free variables, so  $Ax = b$  is either inconsistent or has a unique solution.)

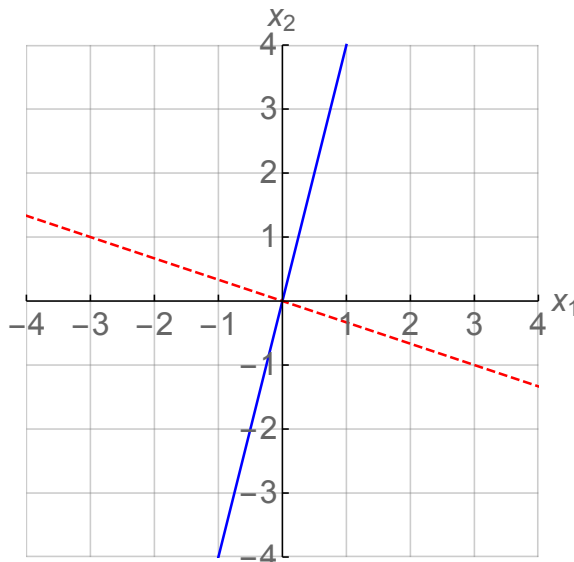
(3) **TRUE** If  $Ax = b$  has a unique solution, then since it is a translation of the solution set to  $Ax = 0$ , this means that  $Ax = 0$  has only the trivial solution (no free variables). Thus,  $A$  has a pivot in every column, which is impossible if  $m < n$  (i.e. impossible if  $A$  has more columns than rows), so  $m \geq n$ .

2. Circle **TRUE** if the statement is always true, and circle **FALSE** otherwise.
- a) If  $A$  is a  $3 \times 10$  matrix with 2 pivots, then  $\dim(\text{Nul}A) = 8$  and  $\text{rank}(A) = 2$ .  
**TRUE**      **FALSE**
- b) If  $A$  is an  $m \times n$  matrix and  $Ax = 0$  has only the trivial solution, then the transformation  $T(x) = Ax$  must have  $\mathbf{R}^m$  as its range.  
**TRUE**      **FALSE**
- c) If  $\{a, b, c\}$  is a basis of a subspace  $V$ , then  $\{a, a + b, b + c\}$  is a basis of  $V$  as well.  
**TRUE**      **FALSE**

**Solution.**

- a) True. Recall that when we say a matrix has two pivots, we mean that its RREF has two pivots.  $\text{rank}(A)$  is the same as number of pivots in  $A$ .  $\dim(\text{Nul}A)$  is the same as the number of free variables. Moreover by the Rank Theorem,  $\text{rank}(A) + \dim(\text{Nul}A) = 10$ , so  $\dim(\text{Nul}A) = 10 - 2 = 8$ .
- b) False. For example,  $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$  has only the trivial solution for  $Ax = 0$ , but its column space is a 2-dimensional subspace of  $\mathbf{R}^3$ .
- c) True. Because  $a$  and  $b$  are independent,  $a + b$  and  $a$  are linearly independent, and furthermore  $a$  and  $b$  are in  $\text{Span}\{a, a + b\}$ . Next,  $c$  is independent from  $\{a, b\}$ , so  $b + c$  is independent from  $\{a, a + b\}$ , meaning that  $\{a, a + b, b + c\}$  is independent by the increasing span criterion. Since  $a, a + b, b + c$  are all clearly in  $\text{Span}\{a, b, c\}$ , by the basis theorem  $\{a, a + b, b + c\}$  also form a span for  $\text{Span}\{a, b, c\} = V$ . Alternatively, we could notice that  $a, b$ , and  $c$  are  $\text{Span}\{a, a + b, b + c\}$ , and since  $V = \text{Span}\{a, b, c\}$  it is a three-dimensional space spanned by the set of three elements  $\{a, a + b, b + c\}$ , those three elements must form a basis, by the basis theorem.

3. Write a matrix  $A$  so that  $\text{Col}(A)$  is the solid blue line and  $\text{Nul}(A)$  is the dotted red line drawn below.



**Solution.**

We'd like to design an  $A$  with the prescribed column space  $\text{Span}\left\{\begin{pmatrix} 1 \\ 4 \end{pmatrix}\right\}$  and null space  $\text{Span}\left\{\begin{pmatrix} 3 \\ -1 \end{pmatrix}\right\}$ .

We start with analyzing the null space. We can write parametric form of the null space:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} 3 \\ -1 \end{pmatrix} \quad \text{is the same as} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3x_2 \\ x_2 \end{pmatrix}$$

Then this implies the RREF of  $A$  must be  $\begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$ .

Now we need to combine the information that column space is  $\text{Span}\left\{\begin{pmatrix} 1 \\ 4 \end{pmatrix}\right\}$ . That means the second row must be 4 multiple of the first row. Therefore the second row must be  $(4 \ 12)$ . We conclude,

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 12 \end{pmatrix}$$

Note any nonzero scalar multiple of the above matrix is also a solution.

4. Let  $A = \begin{pmatrix} 1 & -5 & -2 & -4 \\ 2 & 3 & 9 & 5 \\ 1 & 1 & 4 & 2 \end{pmatrix}$ , and let  $T$  be the matrix transformation associated to  $A$ , so  $T(x) = Ax$ .

- a) What is the domain of  $T$ ? What is the codomain of  $T$ ? Give an example of a vector in the range of  $T$ .
- b) The RREF of  $A$  is  $\begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ .
- (i) Write bases for  $\text{Col}(A)$  and  $\text{Nul}(A)$ .
- (ii) Is there a vector in the codomain of  $T$  which is not in the range of  $T$ ? Justify your answer.

### Solution.

- a) The domain is  $\mathbf{R}^4$ ; the codomain is  $\mathbf{R}^3$ . The vector  $0 = T(0)$  is contained in the range, as is

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = T \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

- b) (i) First, recall that the columns of  $A$ , which correspond to pivots in the RREF of  $A$ , form a basis for  $\text{Col}(A)$ . We note that the first and second column of the RREF of  $A$  contain pivots. Therefore, the first and second columns of  $A$  form a basis for  $\text{Col}(A)$ . That is,

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix} \right\}.$$

Notice that the columns in the RREF of  $A$  do not form such basis themselves and in order to write a basis for  $\text{Col}(A)$ , we need to use the corresponding columns in the matrix  $A$  itself.

In order to write a basis for  $\text{Nul}(A)$ , we need to find the solution to the matrix equation  $A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$  in parametric vector form. Since  $x_3$  and  $x_4$  are free variables in this example. The parametric solution would be

$$\begin{cases} x_1 = -3x_3 - x_4 \\ x_2 = -x_3 - x_4 \end{cases},$$

which in vector form can be written as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, a basis for  $\text{Nul}(A)$  is given by

$$\left\{ \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(ii) Yes. The range of  $T$  is the column span of  $A$ , and  $A$  only has two pivots, so its column span is a 2-dimensional subspace of  $\mathbf{R}^3$ . Since  $\dim(\mathbf{R}^3) = 3$ , the range is not equal to  $\mathbf{R}^3$ .