Math 1553 Worksheet §1.2, §1.3

Solutions

1. **a)** Circle the 'operations' that are legal to use in row reduction, in other words, the operations that will not change the solution set of an arbitrary linear system.

(1) $R_2 = R_3 - 3R_2$

(2) $R_3 = 3R_3$

(3) $R_1 \longleftrightarrow R_2$

(4) $R_1 = R_2 - R_3$

(5) $R_2 = R_2 - R_1^2$

(6) $R_3 = R_3 - \sqrt{R_1}$

b) Define "column operations" on augmented matrices in the same fashion as for the row operations, for example multiplying a column by 2. Do column operations preserve the solution set of a linear system? If yes, explain why; if no, give an example when column operations don't preserve the solution set.

Solution.

a) Only (1), (2), (3) are legit operations in row reduction.
(4) $R_1 = R_2 - R_3$ is not because it removed the R_1, and it will lose the information in R_1.
(5), (6) have nonlinear operations R_1^2, $\sqrt{R_1}$.

b) The answer is "NO". For example, let's consider the augmented matrix $\left(\begin{array}{c|c} 1 & 1 \end{array}\right)$ which corresponds to $x_1 = 1$. If you multiply first column of $\left(\begin{array}{c|c} 1 & 1 \end{array}\right)$ by 2, the augmented matrix becomes $\left(\begin{array}{c|c} 2 & 1 \end{array}\right)$, which means $2x_1 = 1$ so $x_1 = \frac{1}{2}$.

2. **a)** Which of the following matrices are in row echelon form (REF)? Which are in reduced row echelon form (RREF)?

b) For the matrices that are in REF or RREF, which entries are the pivots? What are the pivot columns?

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right) \qquad \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

c) Why is RREF useful, i.e. what information does it reveal about the linear system?

d) How many nonzero entries are there in a pivot column of a matrix that is in RREF?

Solution.

a) The first is in reduced row echelon form; the second is in row echelon form. The third is neither.

b) The pivots are in red; the other entries in the pivot columns are in blue. The third is not in REF, but with one swap $R_2 \longleftrightarrow R_3$ it will be REF and pivots are easy to find.

c) One reason why RREF is useful is that it tells us whether a system is consistent. Namely, if the augmented matrix's RREF has a pivot in the rightmost column, then the system is inconsistent; if not, then it is consistent.

d) In a pivot column of RREF, we will have to clear all entries above and below the pivot. This means it has only 1 nonzero entry.

3. Each matrix below is in RREF. In each case, determine whether the corresponding system of linear equations is consistent, and if so, write the parametric form of the general solution and state how many solutions the system has.

$$\text{(a)} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right), \quad \text{(b)} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right), \quad \text{(c)} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right), \quad \text{(d)} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Solution.

a) [1 solution]. $x_1 = 0, \quad x_2 = 0, \quad x_3 = 1.$

b) [no solution]. There is a pivot in the rightmost column.

c) [infinitely many solutions]. x_3 is a free variable.
$x_1 = 0, \quad x_2 = -2x_3, \quad x_3 = x_3$ (x_3 real), $\quad x_4 = 5.$

d) [infinitely many solutions]. All three variables are free.
$x_1 = x_1, \quad x_2 = x_2, \quad x_3 = x_3 \quad (x_1, x_2, x_3$ real).

4. Find the parametric form for the solution set of the following system of linear equations in x_1, x_2, and x_3 by putting an augmented matrix into reduced row echelon form. State which variables (if any) are free variables. Describe the solution set geometrically.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ - 3x_2 - 6x_3 &= -3. \end{aligned}$$

Solution.

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{array}\right) \xrightarrow{R_2=R_2+4R_1} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & -3 & -6 & -3 \end{array}\right)$$

$$\xrightarrow{R_3=R_3+R_2} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1=R_1-R_2} \left(\begin{array}{ccc|c} 1 & 0 & -5 & -2 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_2=R_2\div 3} \left(\begin{array}{ccc|c} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The variables x_1 and x_2 correspond to pivot columns, but x_3 is free.

$$x_1 = -2 + 5x_3, \qquad x_2 = 1 - 2x_3, \qquad x_3 = x_3 \quad (x_3 \text{ real}).$$

This consistent system in three variables has one free variable, so the solution set is a line in $\mathbf{R}^3$.