MATH 1553, FALL 2023
SAMPLE MIDTERM 3B: COVERS 3.5 THROUGH 5.5

Name		GT ID	

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 70 points, and you have 75 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and all work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. We recommend completing the practice exam in 75 minutes, without notes or distractions.

The exam is not designed to test material from the previous midterm on its own. However, knowledge of the material prior to section §3.5 is necessary for everything we do for the rest of the semester, so it is fair game for the exam as it applies to §§3.5 through 5.5.

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Problem 1.

True or false. Circle **T** if the statement is *always* true.
Otherwise, circle **F**. You do not need to show work or justify your answer.

a) **T** **F** If A is an $n \times n$ matrix, then the determinant of A is the same as the determinant of the RREF of A.

b) **T** **F** If A is a 3×3 matrix with characteristic polynomial

$$\det(A - \lambda I) = (1-\lambda)(-1-\lambda)^2,$$

then A must be invertible.

c) **T** **F** Suppose A is an $n \times n$ matrix and λ is an eigenvalue of A. If v and w are two different eigenvectors of A corresponding to the eigenvalue λ, then $v - w$ is an eigenvector of A.

d) **T** **F** If A and B are 3×3 matrices that have the same eigenvalues and the same algebraic multiplicity for each eigenvalue, then $A = B$.

e) **T** **F** Suppose A is an $n \times n$ matrix and the matrix transformation T given by $T(x) = Ax$ is onto. Then T must also be one-to-one.

Problem 2.

You do not need to show your work or justify your answers.

a) Complete the following definition (be mathematically precise!):
Suppose A is an $n \times n$ matrix and λ is a real number. We say λ is an *eigenvalue* of A if...

b) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 3.$ Find $\det\begin{pmatrix} -4a+d & -4b+e & -4c+f \\ a & b & c \\ g & h & i \end{pmatrix}$.

c) Write a 2×2 matrix which is neither diagonalizable nor invertible.

d) Suppose A is an $n \times n$ matrix and $\det(A) = 0$.
Which of the following statements must be true? Circle all that apply.

(i) $\dim(\mathrm{Nul}(A)) \geq 1$.

(ii) The equation $Ax = 0$ has only the trivial solution $x = 0$.

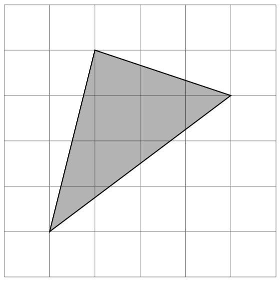
(iii) $\lambda = 0$ is an eigenvalue of A.

(iv) The equation $Ax = b$ must be inconsistent for at least one b in $\mathbf{R}^n$.

Problem 3.

Short answer. You do not need to show your work unless you are told to do so.

a) Find the area of the triangle given below (the grid marks are spaced one unit apart). Briefly show your work.



b) Suppose A is a 3×3 matrix and its characteristic polynomial is

$$\det(A - \lambda I) = -(\lambda - 5)(\lambda - 3)^2.$$

Which of the following *must* be true? Circle all that apply.

(i) The 5-eigenspace of A has dimension 1.

(ii) If the 3-eigenspace of A is the xy-plane, then A is diagonalizable.

(iii) $\det(A) = 45$.

(iv) The homogeneous system given by the equation $(A - 3I)x = 0$ has two free variables.

c) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation that reflects across the line $y = -2x$, and let A be the standard matrix for T. Which of the following are true? Circle all that apply.

(i) The 1-eigenspace of A is $\mathrm{Span}\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}$.

(ii) A is diagonalizable.

(iii) $\det(A + I) = 0$.

Problem 4.

Short answer. Show your work on part (c).

a) Suppose A is a 2×2 matrix and $A^{-1} = \begin{pmatrix} 5 & 4 \\ 2 & 2 \end{pmatrix}$. Find A.

b) Suppose $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -1/3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}^{-1}$. Which of the following are true? Circle all that apply.

(i) Every nonzero vector in $\mathbf{R}^2$ is an eigenvector of A.

(ii) Repeated multiplication by A pushes vectors toward $\text{Span}\left\{\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right\}$.

(iii) If $x = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, then $A^n x$ approaches the zero vector as n becomes very large.

(iv) The eigenvalues of A are $-\dfrac{1}{3}$ and 1.

c) Suppose A is an $n \times n$ matrix. Which of the following statements guarantee that A is invertible? Clearly circle all that apply.

(i) For every y in $\mathbf{R}^n$, the equation $Ax = y$ is consistent.

(ii) For some b in $\mathbf{R}^n$, the equation $Ax = b$ has exactly one solution.

(iii) The RREF of A has a pivot in every column except the rightmost column.

Problem 5.

Parts (a) and (b) are unrelated.

a) Let $A = \begin{pmatrix} 5 & 5 \\ -2 & -1 \end{pmatrix}$. Find the complex eigenvalues of A. For the eigenvalue with positive imaginary part, find one corresponding eigenvector.

b) Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 2 & c & c & 1 \\ 3 & 0 & 0 & 4 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Find all values of c so that $\det(A) = 4$.

Problem 6.

Let $A = \begin{pmatrix} 3 & 0 & -2 \\ 2 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$.

a) Find all eigenvalues of A.

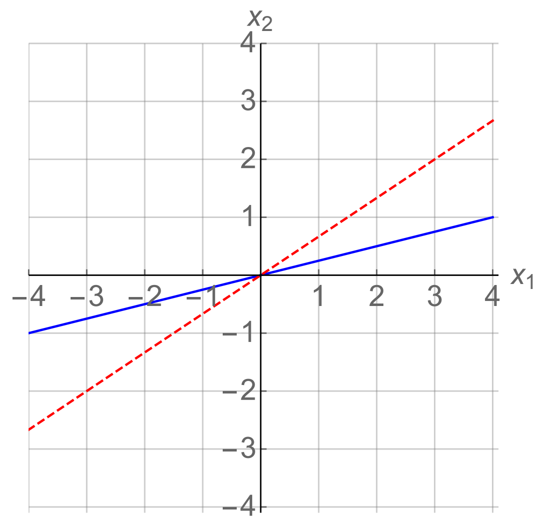
b) Find a basis for each eigenspace of A.

c) Is A diagonalizable? If so, write an invertible 3×3 matrix C and a diagonal matrix D so that $A = CDC^{-1}$. If not, justify why A is not diagonalizable.

Problem 7.

Parts (a), (b), and (c) are unrelated.

a) Consider the matrix A whose 1-eigenspace is the solid blue line below and whose 2-eigenspace is the dotted red line below. Find $A\begin{pmatrix}7\\3\end{pmatrix}$.



b) Suppose A and B are 4×4 matrices satisfying

$$\det(A) = 5, \quad \det(AB^{-1}) = 10.$$

Find $\det(-2B)$. Simplify your answer completely.

c) Find all values of c so that the following matrix has exactly one real value of algebraic multiplicity 2.

$$B = \begin{pmatrix} 2 & c \\ -c & 8 \end{pmatrix}.$$

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.