MATH 1553, FALL 2023
SAMPLE MIDTERM 3A: COVERS 3.5 - 5.5

Name		**GT ID**	

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 70 points, and you have 75 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and all work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. We recommend completing the practice exam in 75 minutes, without notes or distractions.

The exam is not designed to test material from the previous midterm on its own. However, knowledge of the material prior to section §3.5 is necessary for everything we do for the rest of the semester, so it is fair game for the exam as it applies to §§3.5 through 5.5.

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Problem 1.

For each statement, answer TRUE or FALSE. If the statement is *ever* false, circle FALSE. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

a) Suppose S is a rectangle in $\mathbf{R}^2$ with area 2, and let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the matrix transformation

$$T(x) = \begin{pmatrix} 1 & 0 \\ 15 & 3 \end{pmatrix} x.$$

Then the area of $T(S)$ is 6.

$\boxed{\text{TRUE}}$ FALSE

b) If A is an $n \times n$ matrix and the equation $Ax = b$ has at least one solution for each b in $\mathbf{R}^n$, then A must be invertible.

$\boxed{\text{TRUE}}$ FALSE

c) There is an $n \times n$ matrix A so that the zero vector is an eigenvector of A.

TRUE $\boxed{\text{FALSE}}$

d) Let $A = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}$. Then $\lambda = 0$ is an eigenvalue of A.

$\boxed{\text{TRUE}}$ FALSE

e) Let A be the 2×2 matrix that rotates vectors in $\mathbf{R}^2$ by 65 degrees counterclockwise. Then A has no real eigenvalues. $\boxed{\text{TRUE}}$ FALSE

Solution.

a) True: the area is $2|\det(A)| = 2(1(3) - 15(0)) = 6$. This problem is essentially #9 in the Webwork for Determinants I.

b) True: if A is an $n \times n$ matrix whose corresponding matrix transformation is onto, then A is invertible by the Invertible Matrix Theorem.

c) False: the zero vector is **never** an eigenvector of any matrix. This is a fundamental emphasis of Math 1553.

d) True: one row-replacement shows A is not invertible because it has only two pivots.

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix} \xrightarrow{R_3 = R_3 + R_1} \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The fact that A is not invertible is precisely the same thing as saying that $Ax = 0x$ has infinitely many solutions, so $\lambda = 0$ is an eigenvalue of A.

e) True: if x is in $\mathbf{R}^2$ and not the zero vector, then x and Ax are on different lines through the origin, so neither is a scalar multiple of the other.

Problem 2.

Parts (a), (b), (c), and (d) are unrelated. On (a) and (b), you do not need to show your work, and there is no partial credit. Show your work on (c) and (d).

a) (2 points) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 7$. Find $\det\begin{pmatrix} d & e & f \\ a & b & c \\ -2a+g & -2b+h & -2c+i \end{pmatrix}$.

Clearly circle the correct answer below.

(i) 7 (ii) -7 (iii) 14 (iv) -14

(v) 56 (vi) -56 (vii) not enough information (viii) none of these

b) (2 points) Let $A = \begin{pmatrix} 1 & -4 \\ 3 & 15 \end{pmatrix}$. What is A^{-1}? Select the correct choice below.

(i) $A^{-1} = \frac{1}{3}\begin{pmatrix} 15 & -4 \\ 3 & 1 \end{pmatrix}$ (ii) $A^{-1} = \frac{1}{3}\begin{pmatrix} 15 & 4 \\ -3 & 1 \end{pmatrix}$ (iii) $A^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 4 \\ -3 & 15 \end{pmatrix}$

(iv) $A^{-1} = \frac{1}{27}\begin{pmatrix} 15 & -4 \\ 3 & 1 \end{pmatrix}$ $\boxed{\text{(v)}}$ $A^{-1} = \frac{1}{27}\begin{pmatrix} 15 & 4 \\ -3 & 1 \end{pmatrix}$ (vi) $A^{-1} = \frac{1}{27}\begin{pmatrix} 1 & 4 \\ -3 & 15 \end{pmatrix}$

c) (3 points) Find the area of the triangle with vertices $(1,2)$, $(2,3)$, and $(4,-5)$.

d) (3 points) Find all values of a so that $\det\begin{pmatrix} 1 & -3 & 0 \\ 1 & a & 0 \\ a & 0 & a \end{pmatrix} = 0$.

Solution.

a) This problem was nearly copied from #7 in the Webwork for Determinants I. The final matrix is obtained from the original matrix by swapping the first two rows (multiplying the determinant by -1) and then doing the row replacement operation of adding -2(second row) to the third row (does not change the determinant). Therefore, the determinant of the final matrix is $7(-1) = -7$.

b) This is a standard example with inverses, similar to #1 in the 3.5-3.6 Webwork. It is also an easier version of #2d from Sample Midterm 2B. The answer is (v): $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ gives $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, so

$$A^{-1} = \frac{1}{1(15)-(-4)(3)}\begin{pmatrix} 15 & 4 \\ -3 & 1 \end{pmatrix} = \frac{1}{27}\begin{pmatrix} 15 & 4 \\ -3 & 1 \end{pmatrix}.$$

This problem is completely doable without knowing the 2×2 inverse formula. All you need is to understand that A^{-1} is the matrix that satisfies $AA^{-1} = I$. By trial and error, you can multiply A by each of the matrices and quickly determine which one is A^{-1}. In fact, almost all choices except (v) immediately give you something other than 1 in the "11" entry if you multiply them by A, so you can eliminate the other options pretty quickly.

c) This problem is #8 in the Webwork for Determinants I, with changed numbers. We can choose our favorite starting vertex and form vectors v_1 and v_2 from that vertex to the others.

v_1 from $(1,2)$ to $(2,3)$: $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

v_2 from $(1,2)$ to $(4,-5)$: $v_2 = \begin{pmatrix} 3 \\ -7 \end{pmatrix}$.

$$\text{Area} = \frac{1}{2}\left|\det\begin{pmatrix} 1 & 3 \\ 1 & -7 \end{pmatrix}\right| = \frac{1}{2}|-7-3| = \frac{1}{2}(10) = 5.$$

d) This problem was taken from #1 in the Webwork for Determinants II, with changed numbers. We need $1(a^2-0)-(-3)(a-0)+0=0$, so $a^2+3a=a(a+3)=0$. Therefore, $a=0$ and $a=-3$ are the values that make the determinant equal to 0.

Problem 3.

Parts (a), (b), and (c) are unrelated. You do not need to show your work on this page, and there is no partial credit.

a) (3 points) Let A and B be 3×3 matrices satisfying $\det(A) = 2$ and $\det(B) = -3$. Which of the following must be true? Clearly circle all that apply.

(i) $\det(A+B) = \det(A) + \det(B)$.

$\boxed{\text{(ii)}}$ $\det(A^T B^{-1}) = -2/3$.

$\boxed{\text{(iii)}}$ $\det(-2A) = -16$.

b) (4 points) Suppose A is an $n \times n$ matrix. Which of the following conditions guarantee that $\lambda = 4$ is an eigenvalue of A? Clearly circle all that apply.

$\boxed{\text{(i)}}$ The equation $(A - 4I)x = 0$ has infinitely many solutions.

(ii) There is a nonzero vector x in $\mathbf{R}^n$ so that the set $\{x, Ax\}$ is linearly dependent.

$\boxed{\text{(iii)}}$ There is a non-trivial solution to the equation $Ax = 4x$.

(iv) $\mathrm{Nul}(A - 4I) = \{0\}$.

c) (3 points) Suppose A is a 3×3 matrix with characteristic polynomial

$$\det(A - \lambda I) = -\lambda(\lambda + 1)^2.$$

Which of the following statements are true? Clearly circle all that apply.

$\boxed{\text{(i)}}$ The eigenvalues of A are -1 and 0.

(ii) A cannot be diagonalizable.

$\boxed{\text{(iii)}}$ The null space of A must be 1-dimensional.

Solution.

a) We took (i) directly from #8 in the Webwork for Determinants II. It is rarely true that $\det(A+B) = \det(A) + \det(B)$, for example if $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$

then $\det(A)+\det(B)=1$ but $\det(A+B)=-6$.
(ii) is true, since

$$\det(A^T B^{-1}) = \det(A^T)\det(B^{-1}) = \det(A)\cdot\frac{1}{\det(B)} = 2\cdot\frac{-1}{3} = -2/3.$$

(iii) is true since

$$\det(-2A) = (-2)^3\det(A) = -8(2) = -16.$$

b) (i) and (iii) are true by the definition of eigenvalue, but (ii) is not necessarily true (it just means A has a real eigenvalue), and (iv) is the statement that 4 is NOT an eigenvalue.

c) (i) is true since $\lambda = 0$ and $\lambda = -1$ are the values of λ that satisfy the characteristic equation.
(ii) is not necessarily true because A might be diagonalizable, for example

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(iii) is true and is a classic fact: the algebraic multiplicity of $\lambda = 0$ is 1, therefore the geometric multiplicity of $\lambda = 0$ (aka the dimension of the 0-eigenspace, aka the dimension of the null space of A) must also be 1.

Problem 4.

Parts (a), (b), and (c) are unrelated.

a) Let A be the 3×3 matrix for projection onto the xy-plane in $\mathbf{R}^3$.
(i) (2 points) What are the eigenvalues of A? The eigenvalues are $\lambda = 0$ and $\lambda = 1$.

(ii) (1 point) Is A diagonalizable? $\boxed{\text{YES}}$ NO

b) (4 points) Let A be the 2×2 matrix that reflects vectors across the line $y = x$. Fill in the blanks below.

One eigenvalue of A is $\lambda_1 = \underline{\quad 1 \quad}$ and an eigenvector for λ_1 is $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

The other eigenvalue of A is $\lambda_2 = \underline{\quad -1 \quad}$ and an eigenvector for λ_2 is $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(others are possible, for example $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ or $v_2 = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ etc.)

c) (3 points) Which of the following statements must be true? Clearly circle all that apply.

(i) If A and B are invertible $n \times n$ matrices, then $(AB)^{-1} = A^{-1}B^{-1}$.

$\boxed{\text{(ii)}}$ An $n \times n$ matrix A is not invertible if one of its columns is a linear combination of its other columns.

$\boxed{\text{(iii)}}$ If an $n \times n$ matrix A is invertible, then its reduced row echelon form is I_n (the $n \times n$ identity matrix).

Solution.

a) (i) The eigenvalues are $\lambda = 0$ and $\lambda = 1$, because $Av = 0$ for all v along the z-axis and $Av = v$ for all v in the xy-plane. These eigenvalues already give 3 linearly independent eigenvectors in $\mathbf{R}^3$, so there cannot be any more eigenvalues.

(ii) Yes. In fact, A is already *diagonal*! $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

b) One eigenvalue is $\lambda_1 = 1$, and one possibility for v_1 is $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ since A fixes all vectors along the line $y = x$.

$\lambda_2 = -1$, and one possibility for v_2 is $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ since A flips all vectors along the line through the origin perpendicular to $y = x$ (this is the line $y = -x$).

c) (i) No: the correct general formula is $(AB)^{-1} = B^{-1}A^{-1}$.
(ii) Yes: if the columns of A are linearly dependent, then A is not invertible by the Invertible Matrix Theorem.
(iii) Yes: if A is invertible then its RREF has a pivot in every row and every column, therefore the RREF is I_n.

Problem 5.

Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.

For this problem, let $A = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 4 & 4 \\ 0 & 0 & 3 \end{pmatrix}$.

a) (2 points) Find the eigenvalues of A. You do not need to show your work on this part.
This page is a quintessential diagonalization problem. For part (a), A is upper-triangular, so its eigenvalues are its diagonal entries: $\lambda = 3$ and $\lambda = 4$.

b) (5 points) For each eigenvalue of A, find a basis for the corresponding eigenspace.

$$(A - 3I|0) = \left(\begin{array}{ccc|c} 0 & 1 & 4 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 = R_2 - R_1} \left(\begin{array}{ccc|c} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This gives x_1 free, $x_2 = -4x_3$, and x_3 free.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ -4x_3 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix}. \quad \boxed{\text{Basis for 3-eigenspace : } \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix} \right\}}.$$

$$(A - 4I|0) = \left(\begin{array}{ccc|c} -1 & 1 & 4 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right) \xrightarrow{\text{3rd column ops}} \left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This gives $x_1 = x_2$, x_2 free, and $x_3 = 0$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \boxed{\text{Basis for 4-eigenspace : } \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}}.$$

c) (3 points) The matrix A is diagonalizable. Write a 3×3 matrix C and a 3×3 diagonal matrix D so that $A = CDC^{-1}$. Enter your answer below.

We form C using linearly independent eigenvectors and form D using the eigenvalues written **in the corresponding order**. Many answers are possible. For example,

$$C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

or

$$C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -4 \\ 0 & 0 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Problem 6.

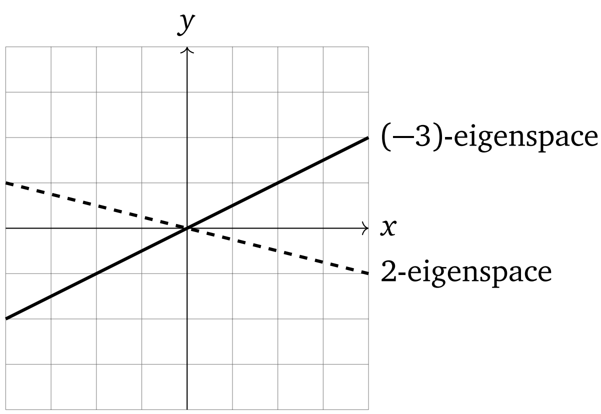
Free response. Fully simplify your answers. Parts (a) and (b) are unrelated. Show your work! A correct answer without sufficient work may receive little or no credit.

a) Let $A = \begin{pmatrix} 1 & -5 \\ 2 & 3 \end{pmatrix}$.

(i) (3 points) Find the complex eigenvalues of A.

(ii) (3 pts) For the eigenvalue with positive imaginary part, find an eigenvector v.

b) (4 points) Let A be the 2×2 matrix whose (-3)-eigenspace is the **solid** line below and whose 2-eigenspace is the **dashed** line below.



Find $A\begin{pmatrix} 6 \\ 0 \end{pmatrix}$. Enter your answer here: $A\begin{pmatrix} 6 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$.

Solution.

a) (i) The characteristic polynomial is

$$\det(A - \lambda I) = (\lambda - 1)(\lambda - 3) - (2)(-5) = \lambda^2 - 4\lambda + 13.$$

The eigenvalues are $\lambda = \dfrac{4 \pm \sqrt{(-4)^2 - 4(13)(1)}}{2} = \dfrac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i$.

(ii) For $\lambda = 2 + 3i$, we use the 2×2 trick: If the first row of $A - \lambda I$ is $\begin{pmatrix} a & b \end{pmatrix}$ (where a and b are not both zero), then $\begin{pmatrix} -b \\ a \end{pmatrix}$ is an eigenvector of A.

$$\left(A - (2+3i)I \mid 0\right) = \left(\begin{array}{cc|c} -1-3i & -5 & 0 \\ (*) & (*) & 0 \end{array}\right),$$

so $v = \begin{pmatrix} 5 \\ -1-3i \end{pmatrix}$ is an eigenvector. An equivalent answer is $\begin{pmatrix} -5 \\ 1+3i \end{pmatrix}$ which is just -1*(the previous answer for v).

b) The graph says that (-3)-eigenspace is Span$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and the 2-eigenspace is Span$\begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

We can solve or just observe $\begin{pmatrix}6\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}4\\-1\end{pmatrix}$, so

$$\begin{aligned} A\begin{pmatrix}6\\0\end{pmatrix} &= A\left(\begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}4\\-1\end{pmatrix}\right) = A\begin{pmatrix}2\\1\end{pmatrix} + A\begin{pmatrix}4\\-1\end{pmatrix} \\ &= -3\begin{pmatrix}2\\1\end{pmatrix} + 2\begin{pmatrix}4\\-1\end{pmatrix} = \begin{pmatrix}-6\\-3\end{pmatrix} + \begin{pmatrix}8\\-2\end{pmatrix} = \begin{pmatrix}2\\-5\end{pmatrix}. \end{aligned}$$

Problem 7.

Free response. Show your work! A correct answer without sufficient work may receive little or no credit. Parts (a), (b), and (c) are unrelated.

a) (3 points) Write a 2×2 matrix A that satisfies both of the following conditions:
(i) A is not diagonalizable
(ii) A has exactly one real eigenvalue.
Justify why your matrix A satisfies both conditions.

Many examples are possible, for example

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

It is upper-triangular so its only eigenvalue is the repeated diagonal $\lambda = 1$. However, A is not diagonalizable because its 1-eigenspace is the null space of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, which is only 1-dimensional.

b) (4 points) Find all values of c so that $\lambda = 2$ is an eigenvalue of $\begin{pmatrix} 3 & c \\ 2 & 1 \end{pmatrix}$.

Here $A = \begin{pmatrix} 3 & c \\ 2 & 1 \end{pmatrix}$ and we need $\lambda = 2$ to be an eigenvalue, which means $A - 2I$ is not invertible. We row-reduce

$$\left(A - 2I \mid 0\right) = \left(\begin{array}{cc|c} 1 & c & 0 \\ 2 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & c & 0 \\ 0 & -1-2c & 0 \end{array}\right).$$

Since $A - 2I$ is not invertible, we have $-1 - 2c = 0$, so $c = -1/2$. Alternatively, we could have solved for $\det(A - 2I) = 0$ and found $c = -1/2$.

c) (3 points) Find $\det \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 1 & -1 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 6 & 0 & 1 \end{pmatrix}$.

Cofactor expansion along the third row gives

$$\begin{aligned} 4(-1)^{3+4} \det \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & -1 \\ 0 & 6 & 0 \end{pmatrix} &= -4 \det \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & -1 \\ 0 & 6 & 0 \end{pmatrix} \\ &= -4 \cdot 6(-1)^{3+2} \det \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \\ &= (-4)(-6)(-1+2) = 24. \end{aligned}$$

Alternatively, we could have done the standard 3×3 determinant formula after our initial cofactor expansion.

$$\begin{aligned}
4(-1)^{3+4} \det \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & -1 \\ 0 & 6 & 0 \end{pmatrix} &= -4 \det \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & -1 \\ 0 & 6 & 0 \end{pmatrix} \\
&= -4\Big[1(6) - 2(0) - 1(12)\Big] \\
&= -4(-6) = 24.
\end{aligned}$$

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.