MATH 1553, FALL 2023
SAMPLE MIDTERM 2B: COVERS 2.5 - 3.4

Name		**GT ID**	

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 70 points, and you have 75 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and all work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. We recommend completing the practice exam in 75 minutes, without notes or distractions.

The exam is not designed to test material from the previous midterm on its own. However, knowledge of the material prior to section §2.5 is necessary for everything we do for the rest of the semester, so it is fair game for the exam as it applies to §§2.5 through 3.4.

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Problem 1.

True or false. Circle **T** if the statement is *always* true.
Otherwise, circle **F**. You do not need to show work or justify your answer, and there is no partial credit.

a) **T** **F** If $\{v_1, \ldots, v_p\}$ is a linearly independent set of vectors in $\mathbf{R}^n$, then $p \leq n$.

b) **T** **F** There is a 4×7 matrix A that satisfies $\dim(\text{Nul}A) = 1$.

c) **T** **F** If $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation, then the zero vector must be a solution to the equation $T(x) = 0$.

d) **T** **F** Suppose A is an $n \times n$ matrix and $Ax = 0$ has only the trivial solution. Then each b in $\mathbf{R}^n$ can be written as a linear combination of the columns of A.

e) **T** **F** Suppose v_1, v_2, v_3, v_4 are vectors in $\mathbf{R}^5$, so that $\text{Span}\{v_1, v_2\}$ has dimension 2 and $\text{Span}\{v_3, v_4\}$ has dimension 2. Then $\text{Span}\{v_1, v_2, v_3, v_4\}$ has dimension 4.

Solution.

a) True. If $p > n$, then the vectors $\{v_1, \ldots, v_p\}$ in $\mathbf{R}^n$ are automatically linearly dependent.

b) False. If $\dim(\text{Nul } A) = 1$ then $\dim(\text{Col}A) = 6$ which is impossible since Col A is a subspace of $\mathbf{R}^4$.

c) True. One way to see this is that $T(x) = Ax$ for some $m \times n$ matrix A, and $A0 = 0$.

d) True by a pivot counting argument: we were given that our matrix A is $n \times n$ with a pivot in every column, thus A has n pivots, thus A has a pivot in every row so T is onto.

e) False. For example we could have $v_1 = v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $v_2 = v_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Then $\dim(\text{Span}\{v_1, v_2, v_3, v_4\}) = 2$.

Problem 2.

You do not need to show your work in parts (a)-(c), but show your work in (d).

a) Let $V = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$.

Fill in the blank: $\dim(V) =$ ____________.

b) Suppose A is an 8×5 matrix, and the range of the transformation $T(x) = Ax$ is a line. Fill in the blank:

Nul(A) is a ________ -dimensional subspace of $\mathbf{R}^{\square}$.

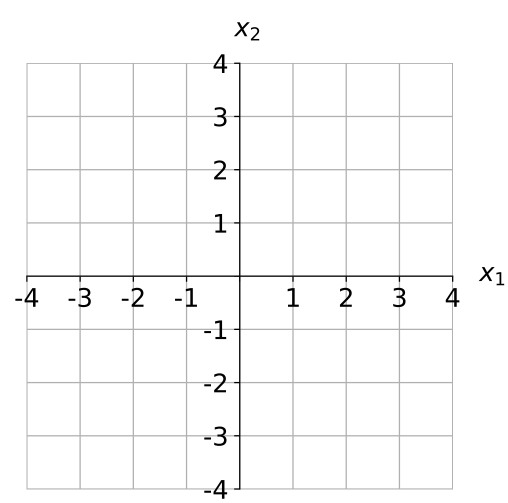
c) Suppose that $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation with standard matrix A. Which of the following conditions *guarantee* that T is one-to-one? Circle all that apply.

(i) For each x in $\mathbf{R}^n$, there is a unique y in $\mathbf{R}^m$ so that $T(x) = y$.

(ii) For each y in $\mathbf{R}^m$, the matrix equation $Ax = y$ is consistent.

(iii) The columns of A are linearly independent.

d) Write a 3×2 matrix A whose column space is a line, then draw the **null space** of A below.



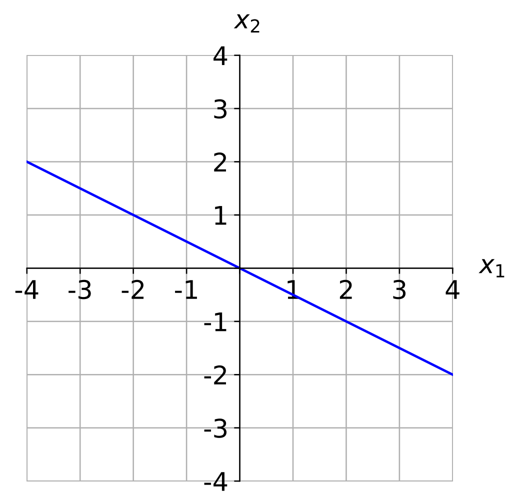
Solution.

a) $\dim(V) = 2$ since the first two vectors span V and are linearly independent.

b) $\dim(\text{Nul } A) = 5 - \dim(\text{Col} A) = 5 - 1 = 4$, and Nul A is a subspace of $\mathbf{R}^5$.

c) Only (iii). Statement (i) is just the definition of a transformation, and (ii) defines onto.

d) One such matrix is $A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$. Its null space is the line $x_1 = -2x_2$, which is the span of $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.



Problem 3.

Short answer and multiple choice. You do not need to show your work, and there is no partial credit.

a) (i) Write the matrix A for the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ that reflects vectors across the line $y = x$.

(ii) Write the matrix B for the linear transformation $U : \mathbf{R}^2 \to \mathbf{R}^2$ that reflects vectors across the x-axis.

b) In each case, determine whether the given set of vectors is linearly dependent or linearly dependent. Clearly circle your answer.

(i) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\}.$

Linearly Dependent Linearly independent

(ii) $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \right\}.$

Linearly Dependent Linearly independent

(iii) $\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \end{pmatrix} \right\}.$

Linearly Dependent Linearly independent

c) Let $A = \begin{pmatrix} 1 & -3 & 0 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix}$, and let T be the matrix transformation $T(x) = Ax$.

(i) What is the domain of T? $\mathbf{R}^4$

(ii) What is the codomain of T? $\mathbf{R}^3$

(iii) Is T one-to-one? no

(iv) Is T onto? yes

Solution.

a) (i) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; (ii) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

b) (i) Linearly dependent. It is a set of 4 vectors in $\mathbf{R}^3$, so without any work we know it is a linearly dependent set.
(ii) Linearly dependent. The two vectors are scalar multiples of each other.
(iii) Linearly independent.

c) A is a 3×4 matrix with 3 pivots.
(i) The domain of T is $\mathbf{R}^4$.
(ii) The codomain of T is $\mathbf{R}^3$.
(iii) No, because A does not have a pivot in every column.
(iv) Yes, because A has a pivot in every row.

Problem 4.

Parts (a), (b), and (c) are unrelated. Show your work briefly in parts (b) and (c).

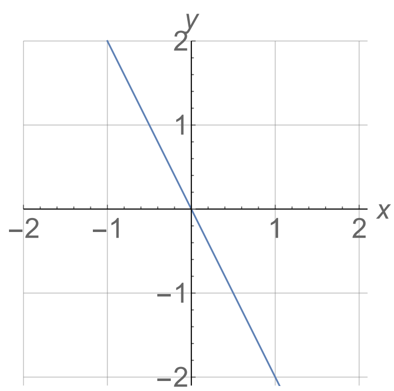
a) Write a matrix A whose null space is the line $y = x$ in $\mathbf{R}^2$ and whose matrix transformation $T(x) = Ax$ has range equal to the z-axis in $\mathbf{R}^3$.

b) Let $A = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 0 & -4 \\ -4 & 0 & 8 \end{pmatrix}$.

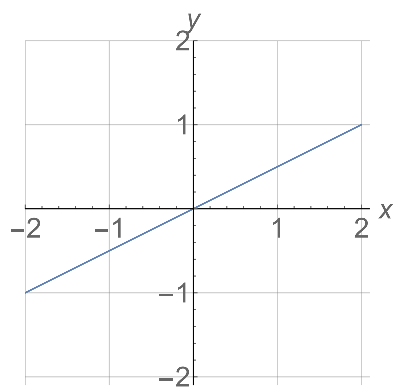
Write nonzero vectors x and y in $\mathbf{R}^3$ so that $Ax = Ay$ but $x \neq y$.

c) Let $A = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$. Clearly draw Col($A$) and Nul($A$).

Draw Col(A) here.



Draw Nul(A) here.



Solution.

a) We need Col(A) to be the span of $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. We satisfy the condition for the null space if solving $(A|0)$ gives us $x - y = 0$ (y free), so an RREF of $\begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ will work.

Many examples are possible. For example, $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{pmatrix}$ or $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -4 & 4 \end{pmatrix}$.

b) We find two nonzero vectors in Nul$A$. $\left(A \mid 0\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$, so $x_1 = 2x_3$ and x_2 and x_3 are free. We can take $x = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, as $Ax = Ay = 0$.

c) Col(A) is the span of $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$; Nul($A$) is the span of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Problem 5.

Show your work on parts (a), (b), and (f).

Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be the transformation $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y - z \\ x + 2y \end{pmatrix}$, and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation of rotation counterclockwise by 90°.

a) Write the standard matrix A for T.

b) Write the standard matrix B for U.

c) Is T onto? YES NO

d) Is U one-to-one? YES NO

e) Circle the composition that makes sense: $T \circ U$ $U \circ T$

f) Write the standard matrix for the composition you chose in part (e).

Solution.

a) $A = \begin{pmatrix} T(e_1) & T(e_2) & T(e_3) \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \end{pmatrix}$

b) $B = \begin{pmatrix} \cos(90°) & -\sin(90°) \\ \sin(90°) & \cos(90°) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

c) Yes, T is onto since A has a pivot in each row.

d) Yes, since B has two pivots and is a 2×2 matrix.

e) $U \circ T$ makes sense.

f) $BA = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$.

Problem 6.

Free response. Show your work unless specified otherwise. Parts (a) and (b) are unrelated.

a) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation satisfying

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} \quad \text{and} \quad T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}.$$

Find the standard matrix A for T.

b) William Moreland has put the matrix A below in its reduced row echelon form:

$$A = \begin{pmatrix}2 & 6 & -14 & 7\\ 3 & 9 & -21 & 10\\ 4 & 12 & -28 & 12\end{pmatrix} \xrightarrow{RREF} \begin{pmatrix}1 & 3 & -7 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

(i) Write a basis for Col A (you don't need to justify your answer).

(ii) Write a new basis for Col$A$, so that no vector in your new basis is a scalar multiple of any of the vectors in the basis you wrote in part (i). Clearly show how you obtain this new basis.

(iii) Find one nonzero vector x that satisfies $Ax = 0$.

Solution.

a) $A = \begin{pmatrix} T(e_1) & T(e_2)\end{pmatrix}$ and we're given $T(e_1) = \begin{pmatrix}1\\2\end{pmatrix}$. Using linearity of T and the fact that $\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix}$:

$$T\begin{pmatrix}0\\1\end{pmatrix} = T\begin{pmatrix}1\\1\end{pmatrix} - T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} - \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-2\\-1\end{pmatrix}.$$

Thus $A = \begin{pmatrix}1 & -2\\ 2 & -1\end{pmatrix}$.

b) (i) The pivot columns of A are a basis for Col$A$:

$$\left\{\begin{pmatrix}2\\3\\4\end{pmatrix}, \begin{pmatrix}7\\10\\12\end{pmatrix}.\right\}$$

In reality, the fourth column and any one among the first three columns will form a basis for Col$A$.

(ii) Many possibilities. For example, $\{w_1, w_2\}$ where $w_1 = v_1 - v_2$ and $w_2 = v_1 + v_2$.

$$w_1 = v_1 - v_2 = \begin{pmatrix}-5\\-7\\-8\end{pmatrix} \qquad w_2 = v_1 + v_2 = \begin{pmatrix}9\\13\\16\end{pmatrix}.$$

(iii) Many possibilities. It's not necessary to actually write a basis for Nul$A$ to do this, but any nonzero vector in $\text{Nul}A = \text{Span}\left\{\begin{pmatrix}-3\\1\\0\\0\end{pmatrix}, \begin{pmatrix}7\\0\\1\\0\end{pmatrix}\right\}$ is correct.

Problem 7.

Parts (a) and (b) are unrelated. You don't need to show your work in part (a), but show your work in part (b).

a) Consider the set $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; xy \geq 0 \right\}$.

(i) Does V contain the zero vector? YES NO

(ii) Is V closed under addition? YES NO

(iii) Is V closed under scalar multiplication? YES NO

b) Consider the subspace W of $\mathbf{R}^3$ given by

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x - 5y + 6z = 0 \right\}.$$

(i) Find a basis for W.

(ii) Is there a matrix A so that $\mathrm{Col}(A) = W$? If your answer is yes, write such a matrix A. If your answer is no, justify why there is no such matrix A.

Solution.

a) (i) Yes since $0(0) = 0 \geq 0$.

(ii) No. For example, $\begin{pmatrix} -5 \\ -1 \end{pmatrix}$ is in V and $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ is in V, but

$$\begin{pmatrix} -5 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \text{ which is not in } V \text{ since } (-3)(1) < 0.$$

(iii) Yes. If $\begin{pmatrix} x \\ y \end{pmatrix}$ is in V and c is any real number then $\begin{pmatrix} cx \\ cy \end{pmatrix}$ is in V since $xy \geq 0$ and

$$(cx)(cy) = c^2 xy = c^2(\text{something 0 or greater}) \geq 0.$$

Geometrically, $xy \geq 0$ means $\begin{pmatrix} x \\ y \end{pmatrix}$ is in the first or third quadrant. This region contains 0, is not closed under addition, and is closed under scalar multiplication.

b) (i) $W = \mathrm{Nul}\begin{pmatrix} 1 & -5 & 6 \end{pmatrix}$, so $x = 5y - 6z$ where y and z are free. A basis is

$$\left\{ \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(ii) Certainly. For example,

$$A = \begin{pmatrix} 5 & -6 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.