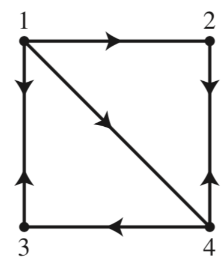
Supplemental problems: §5.6

1. Suppose the internet has four pages in the following manner. Arrows represent links from one page towards another. For example, page 1 links to page 4 but not vice versa.



a) Write the importance matrix and the Google matrix for this internet using damping constant $p = 0.15$. You don't need to simplify the Google matrix.

b) The steady-state vector for the Google matrix is (approximately)

$$\begin{pmatrix} 0.23 \\ 0.23 \\ 0.23 \\ 0.31 \end{pmatrix}.$$

What is the top-ranked page?

Solution.

(a) The importance matrix is

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1 & 0 & 0 \end{pmatrix}$$

The Google matrix is

$$(1-p)A + pB$$

$$0.85 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1 & 0 & 0 \end{pmatrix} + (0.15)\frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

(b) From the steady-state vector we see page 4 has the highest rank.

2. The companies X, Y, and Z fight for customers. This year, company X has 40 customers, Company Y has 15 customers, and Z has 20 customers. Each year, the following changes occur:

- X keeps 75% of its customers, while losing 15% to Y and 10% to Z.
- Y keeps 60% of its customers, while losing 5% to X and 35% to Z.

- Z keeps 65% of its customers, while losing 15% to X and 20% to Y.

Write a stochastic matrix A and a vector x so that Ax will give the number of customers for firms X, Y, and Z (respectively) after one year. You do not need to compute Ax.

Solution.

$$A = \begin{pmatrix} 0.75 & 0.05 & 0.15 \\ 0.15 & 0.6 & 0.20 \\ 0.1 & 0.35 & 0.65 \end{pmatrix} \qquad x = \begin{pmatrix} 40 \\ 15 \\ 20 \end{pmatrix}.$$

3. Suppose p and q are real numbers on the open interval $(0,1)$, and

$$A = \begin{pmatrix} p & 1-q \\ 1-p & q \end{pmatrix}$$

(1) Is A a positive stochastic matrix? Why?
(2) Does A have unique steady state vector? Why?
(3) Without computation, give an eigenvalue of A.
(4) Compute the steady-state vector of A.

Solution.

(1) Yes: columns sum to 1, all entries strictly positive
(2) Yes: A is a positive stochastic matrix, so the Perron-Frobenius theorem applies.
(3) $\lambda = 1$
(4) Solving $(A - I)v = 0$ and scaling v to get the steady-state vector w, we get

$$w = \frac{1}{2-p-q} \begin{pmatrix} 1-q \\ 1-p \end{pmatrix}.$$

Supplemental problems: Chapter 6

1. True or false. If the statement is always true, answer true. Otherwise, answer false. Justify your answer.

a) Suppose $W = \text{Span}\{w\}$ for some vector $w \neq 0$, and suppose v is a vector orthogonal to w. Then the orthogonal projection of v onto W is the zero vector.

b) Suppose W is a subspace of $\mathbf{R}^n$ and x is a vector in $\mathbf{R}^n$. If x is not in W, then $x - x_W$ is not zero.

c) Suppose W is a subspace of $\mathbf{R}^n$ and x is in both W and $W^\perp$. Then $x = 0$.

d) Suppose $\widehat{x}$ is a least squares solution to $Ax = b$. Then $\widehat{x}$ is the closest vector to b in the column space of A.

Solution.

a) True. Since $v \in W^\perp$, its projection onto W is zero.

b) True. If x is not in W then $x \neq x_W$, so $x - x_W$ is not zero.

c) True. Since x is in W and $W^\perp$ it is orthogonal to itself, so $||x||^2 = x \cdot x = 0$. The length of x is zero, which means every entry of x is zero, hence $x = 0$.

d) False: $A\widehat{x}$ is the closest vector to b in Col A.

2. Let $W = \text{Span}\{v_1, v_2\}$, where $v_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

a) Find the closest point w in W to $x = \begin{pmatrix} 0 \\ 14 \\ -4 \end{pmatrix}$.

Let $A = \begin{pmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{pmatrix}$. We solve $A^T A v = A^T x$.

$$A^T A = \begin{pmatrix} 6 & 6 \\ 6 & 14 \end{pmatrix} \qquad A^T \begin{pmatrix} 0 \\ 14 \\ -4 \end{pmatrix} = \begin{pmatrix} 24 \\ 16 \end{pmatrix}.$$

We find $\left(\begin{array}{cc|c} 6 & 6 & 24 \\ 6 & 14 & 16 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -1 \end{array}\right)$, so $v = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$ and therefore

$$w = Av = \begin{pmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -6 \\ 8 \\ 2 \end{pmatrix}.$$

b) Find the distance from w to $\begin{pmatrix} 0 \\ 14 \\ -4 \end{pmatrix}$.

$$||x-w|| = \left\|\begin{pmatrix} 0 \\ 14 \\ -4 \end{pmatrix} - \begin{pmatrix} -6 \\ 8 \\ 2 \end{pmatrix}\right\| = \left\|\begin{pmatrix} 6 \\ 6 \\ -6 \end{pmatrix}\right\| = \sqrt{36+36+36} = \sqrt{108} = 6\sqrt{3}.$$

c) Find the standard matrix for the orthogonal projection onto Span$\{v_1\}$.

$$B = \frac{1}{v_1 \cdot v_1} v_1 v_1^T = \frac{1}{(-1)^2+2^2+1^2}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}\begin{pmatrix} -1 & 2 & 1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1 & -2 & -1 \\ -2 & 4 & 2 \\ -1 & 2 & 1 \end{pmatrix}$$

d) Find the standard matrix for the orthogonal projection onto W.

Let $A = \begin{pmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{pmatrix}$. Since the columns of A are linearly independent, our projection matrix is $A(A^TA)^{-1}A^T$. We already computed A^TA in part (a), so our matrix is

$$A(A^TA)^{-1}A^T = \begin{pmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 6 & 6 \\ 6 & 14 \end{pmatrix}^{-1}\begin{pmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 6 & 6 \\ 6 & 14 \end{pmatrix}^{-1}\begin{pmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

$$= \frac{1}{48}\begin{pmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 14 & -6 \\ -6 & 6 \end{pmatrix}\begin{pmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

3. Find the least-squares line $y = Mx + B$ that approximates the data points

$$(-2,-11), \quad (0,-2), \quad (4,2).$$

Solution.

If there were a line through the three data points, we would have:

$$\begin{aligned} (x=-2) &\qquad B + M(-2) = -11 \\ (x=0) &\qquad B + M(0) = -2 \\ (x=4) &\qquad B + M(4) = 2. \end{aligned}$$

This is the matrix equation $\begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} B \\ M \end{pmatrix} = \begin{pmatrix} -11 \\ -2 \\ 2 \end{pmatrix}$.

Thus, we are solving the least-squares problem to $Av = b$, where

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 4 \end{pmatrix} \qquad b = \begin{pmatrix} -11 \\ -2 \\ 2 \end{pmatrix}.$$

We solve $A^T A\widehat{x} = A^T b$, where $\widehat{x} = \begin{pmatrix} B \\ M \end{pmatrix}$.

$$A^T A = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 20 \end{pmatrix},$$

$$A^T b = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 4 \end{pmatrix} \begin{pmatrix} -11 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ 30 \end{pmatrix}.$$

$$\left(\begin{array}{cc|c} 3 & 2 & -11 \\ 2 & 20 & 30 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 2 & 20 & 30 \\ 3 & 2 & -11 \end{array}\right) \xrightarrow[R_1 = R_1/2]{R_2 = R_2 - \frac{3R_1}{2}} \left(\begin{array}{cc|c} 1 & 10 & 15 \\ 0 & -28 & -56 \end{array}\right) \xrightarrow[R_1 = R_1 - 10R_2]{R_2 = -\frac{R_2}{28}} \left(\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & 2 \end{array}\right).$$

So $\widehat{x} = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$. In other words, $y = -5 + 2x$, or $\boxed{y = 2x - 5}$.