Supplemental problems: §5.4

1. True or false. Answer true if the statement is always true. Otherwise, answer false.

a) If A is an invertible matrix and A is diagonalizable, then A^{-1} is diagonalizable.

b) A diagonalizable $n \times n$ matrix admits n linearly independent eigenvectors.

c) If A is diagonalizable, then A has n distinct eigenvalues.

2. Give examples of 2×2 matrices with the following properties. Justify your answers.

a) A matrix A which is invertible and diagonalizable.

b) A matrix B which is invertible but not diagonalizable.

c) A matrix C which is not invertible but is diagonalizable.

d) A matrix D which is neither invertible nor diagonalizable.

3. $A = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix}$.

a) Find the eigenvalues of A, and find a basis for each eigenspace.

b) Is A diagonalizable? If your answer is yes, find a diagonal matrix D and an invertible matrix C so that $A = CDC^{-1}$. If your answer is no, justify why A is not diagonalizable.

4. Let $A = \begin{pmatrix} 8 & 36 & 62 \\ -6 & -34 & -62 \\ 3 & 18 & 33 \end{pmatrix}$.

The characteristic polynomial for A is $(\lambda-2)^2(\lambda-3)$. Decide if A is diagonalizable. If it is, find an invertible matrix C and a diagonal matrix D such that $A = CDC^{-1}$.

5. Which of the following 3×3 matrices are necessarily diagonalizable over the real numbers? (Circle all that apply.)

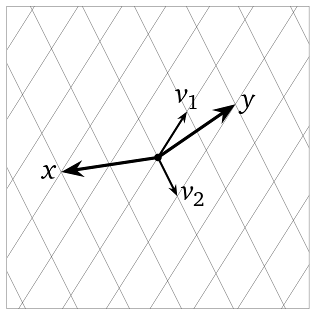
1. A matrix with three distinct real eigenvalues.
2. A matrix with one real eigenvalue.
3. A matrix with a real eigenvalue λ of algebraic multiplicity 2, such that the λ-eigenspace has dimension 2.
4. A matrix with a real eigenvalue λ such that the λ-eigenspace has dimension 2.

6. Suppose a 2×2 matrix A has eigenvalue $\lambda_1 = -2$ with eigenvector $v_1 = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$, and eigenvalue $\lambda_2 = -1$ with eigenvector $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

a) Find A.

b) Find A^{100}.

7. Suppose that $A = C\begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix}C^{-1}$, where C has columns v_1 and v_2. Given x and y in the picture below, draw the vectors Ax and Ay.



Supplemental problems: §5.5

1. **a)** If A is the matrix that implements rotation by $143°$ in $\mathbf{R}^2$, then A has no real eigenvalues.

b) A 3×3 matrix can have eigenvalues $3, 5$, and $2 + i$.

c) If $v = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = 1 - i$, then $w = \begin{pmatrix} 2i-1 \\ i \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = 1 - i$.

2. Consider the matrix

$$A = \begin{pmatrix} 3\sqrt{3}-1 & -5\sqrt{3} \\ 2\sqrt{3} & -3\sqrt{3}-1 \end{pmatrix}$$

a) Find both complex eigenvalues of A.

b) Find an eigenvector corresponding to each eigenvalue.

3. This problem shows an example of a matrix that has a mix of eigenvalues that are real and not real. It isn't computationally feasible on an exam, so doing this problem in full is just for fun. However, understanding the possibilities for eigenvalues of an $n \times n$ matrix in terms of the Fundamental Theorem of Algebra is a key component of section 5.5.

Let $A = \begin{pmatrix} 4 & -3 & 3 \\ 3 & 4 & -2 \\ 0 & 0 & 2 \end{pmatrix}$. Find all eigenvalues of A. For each eigenvalue of A, find a corresponding eigenvector.