Supplemental problems: §3.2

1. Let A be a 3×4 matrix with column vectors v_1, v_2, v_3, v_4, and suppose $v_2 = 2v_1 - 3v_4$. Consider the matrix transformation $T(x) = Ax$.

a) Is it possible that T is one-to-one? If yes, justify why. If no, find distinct vectors v and w so that $T(v) = T(w)$.

b) Is it possible that T is onto? Justify your answer.

Solution.

a) From the linear dependence condition we were given, we get

$$-2v_1 + v_2 + 3v_4 = 0.$$

The corresponding vector equation is just

$$\begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{so} \quad A\begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, $v = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix}$ and $w = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ both satisfy $Av = Aw = 0$, so T cannot be one-to-one.

b) Yes. If $\{v_1, v_3, v_4\}$ is linearly independent then A will have a pivot in every row and T will be onto. Such a matrix A is

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{pmatrix}.$$

2. **a)** Which of the following are onto transformations? (Check all that apply.)

- [x] $T : \mathbf{R}^3 \to \mathbf{R}^3$, reflection over the xy-plane
- [ ] $T : \mathbf{R}^3 \to \mathbf{R}^3$, projection onto the xy-plane
- [x] $T : \mathbf{R}^3 \to \mathbf{R}^2$, project onto the xy-plane, forget the z-coordinate
- [x] $T : \mathbf{R}^2 \to \mathbf{R}^2$, scale the x-direction by 2

b) Let A be a square matrix (square means $n \times n$) and let $T(x) = Ax$. Which of the following guarantee that T is onto? (Check all that apply.)

- [x] T is one-to-one

☐ $Ax = 0$ is consistent

3. Which of the following transformations are **one-to-one**? Circle all that apply.

(i) $T : \mathbf{R}^2 \to \mathbf{R}^3$ defined by $T(x,y) = (x-y,\ x-2y,\ x+y)$.

(ii) $T : \mathbf{R}^2 \to \mathbf{R}^2$ which rotates vectors counterclockwise by 17 degrees.

(iii) The transformation $T(x) = Ax$, where $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$.

Solution.

(i) Yes.
(ii) Yes.
(iii) No.

4. Which of the following transformations are **onto**? Circle all that apply.

(i) $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x,y,z) = (x+y,\ x+y)$.

(ii) $T : \mathbf{R}^2 \to \mathbf{R}^2$ which rotates vectors counterclockwise by 17 degrees.

(iii) The transformation $T(x) = Ax$, where A is a 3×3 matrix with the property that the equation $Av = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ has exactly one solution.

Solution.

(i) No.
(ii) Yes.
(iii) Yes. The condition on A means that $Ax = 0$ has exactly one solution, so A has a pivot in every row.

5. Find all real numbers h so that the transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ given by

$$T(v) = \begin{pmatrix} -1 & 0 & 2-h \\ h & 0 & 3 \end{pmatrix} v$$

is onto.

Solution.

We row-reduce A to find when it will have a pivot in every row:

$$\begin{pmatrix} -1 & 0 & 2-h \\ h & 0 & 3 \end{pmatrix} \xrightarrow{R_2 = R_2 + hR_1} \begin{pmatrix} -1 & 0 & 2-h \\ 0 & 0 & 3+h(2-h) \end{pmatrix}.$$

The matrix has a pivot in every row unless

$$3 + h(2-h) = 0, \qquad h^2 - 2h - 3 = 0, \qquad (h-3)(h+1) = 0.$$

Therefore, T is onto as long as $h \neq 3$ and $h \neq -1$.

Supplemental problems: §3.3

1. Circle **T** if the statement is always true, and circle **F** otherwise.

a) **T** **F** If $T : \mathbf{R}^n \to \mathbf{R}^n$ is linear and $T(e_1) = T(e_2)$, then the homogeneous equation $T(x) = 0$ has infinitely many solutions.

b) **T** **F** If $T : \mathbf{R}^n \to \mathbf{R}^{\mathbf{m}}$ is a one-to-one linear transformation and $m \neq n$, then T must not be onto.

Solution.

a) True. The matrix transformation $T(x) = Ax$ is not one-to-one, so $Ax = 0$ has infinitely many solutions. For example, $e_1 - e_2$ is a non-trivial solution to $Ax = 0$ since $A(e_1 - e_2) = Ae_1 - Ae_2 = 0$.

b) True. Let A be the $m \times n$ standard matrix for T. If T is both one-to-one and onto then T must have a pivot in each column and in each row, which is only possible when A is a square matrix ($m = n$).

2. Consider $T : \mathbf{R}^3 \to \mathbf{R}^3$ given by

$$T(x, y, z) = (x,\ x + z,\ 3x - 4y + z,\ x).$$

Is T one-to-one? Justify your answer.

Solution.

One approach: We form the standard matrix A for T:

$$A = \begin{pmatrix} T(e_1) & T(e_2) & T(e_3) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 3 & -4 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We row-reduce A until we determine its pivot columns

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 3 & -4 & 1 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow[R_3 = R_3 - 3R_1,\ R_4 = R_4 - R_1]{R_2 = R_2 - R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

A has a pivot in every column, so T is one-to-one.

Alternative approach: T is a linear transformation, so it is one-to-one if and only if the equation $T(x, y, z) = (0, 0, 0)$ has only the trivial solution.
If $T(x, y, z) = (x,\ x + z,\ 3x - 4y + z,\ x) = (0, 0, 0, 0)$ then $x = 0$, and

$$x + z = 0 \implies 0 + z = 0 \implies z = 0, \text{ and finally}$$

$$3x - 4y + z = 0 \implies 0 - 4y + 0 = 0 \implies y = 0,$$

so the trivial solution $x = y = z = 0$ is the only solution the homogeneous equation. Therefore, T is one-to-one.

3. Which of the following transformations T are onto? Which are one-to-one? If the transformation is not onto, find a vector not in the range. If the matrix is not one-to-one, find two vectors with the same image.

a) The transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x, y, z) = (y, y)$.

b) JUST FOR FUN: Consider T : (Smooth functions) $\to$ (Smooth functions) given by $T(f) = f'$ (the derivative of f). Then T is not a transformation from any $\mathbf{R}^n$ to $\mathbf{R}^m$, but it is still *linear* in the sense that for all smooth f and g and all scalars c (by properties of differentiation we learned in Calculus 1):

$$T(f+g) = T(f) + T(g) \quad \text{since} \quad (f+g)' = f' + g'$$

$$T(cf) = cT(f) \quad \text{since} \quad (cf)' = cf'.$$

Is T one-to-one?

Solution.

a) This is not onto. Everything in the range of T has its first coordinate equal to its second, so there is no (x, y, z) such that $T(x, y, z) = (1, 0)$. It is not one-to-one: for instance, $T(0, 0, 0) = (0, 0) = T(0, 0, 1)$.

b) T is not one-to-one. If T were one-to-one, then for any smooth function b, the equation $T(f) = b$ would have at most one solution. However, note that if f and g are the functions $f(t) = t$ and $g(t) = t - 1$, then f and g are different functions but their derivatives are the same, so $T(f) = T(g)$. Therefore, T is not one-to-one. It is not within the scope of Math 1553. If you find it confusing, feel free to ignore it.

4. In each case, determine whether T is linear. Briefly justify.

a) $T(x_1, x_2) = (x_1 - x_2, x_1 + x_2, 1)$.

b) $T(x, y) = (y, x^{1/3})$.

c) $T(x, y, z) = 2x - 5z$.

Solution.

a) Not linear. $T(0, 0) = (0, 0, 1) \neq (0, 0, 0)$.

b) Not linear. The $x^{1/3}$ term gives it away. $T(0, 2) = (0, 2^{1/3})$ but $2T(0, 1) = (0, 2)$.

c) Linear. In fact, $T(v) = Av$ where

$$A = \begin{pmatrix} 2 & 0 & -5 \end{pmatrix}.$$

5. The second little pig has decided to build his house out of sticks. His house is shaped like a pyramid with a triangular base that has vertices at the points $(0,0,0)$, $(2,0,0)$, $(0,2,0)$, and $(1,1,1)$.

The big bad wolf finds the pig's house and blows it down so that the house is rotated by an angle of 45° in a counterclockwise direction about the z-axis (look downward onto the xy-plane the way we usually picture the plane as $\mathbf{R}^2$), and then projected onto the xy-plane.

In the worksheet, we found the matrix for the transformation T caused by the wolf. Geometrically describe the image of the house under T.

Solution.

Work shows that $T(x) = Ax$, where

$$A = \begin{pmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We know the house has been effectively destroyed, but what do its remains look like? To get an idea, let's look at what happens to the vertices.

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix} \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix}.$$

This indicates the pyramid has been squashed into a triangle in the xy-plane with vertices $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -\sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix}$. (the point $\begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix}$ is along the top side of this triangle).

Effectively, the pyramid was rotated and then destroyed, so that its (rotated) base is all that remains.