Supplemental problems: §1.2, §1.3

1. True or false.

a) If the RREF of an augmented matrix has a pivot in every column, then the corresponding system of linear equations must be consistent.

b) If the RREF of an augmented matrix has a pivot in every column except its rightmost column, then the corresponding system of linear equations has exactly one solution.

c) If the RREF of an augmented matrix has final row $\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \end{array}\right)$, then the corresponding system of linear equations has infinitely many solutions.

Solution.

a) False. The system will be inconsistent because there is a pivot in the rightmost column, for example $\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$.

b) True. This means the system is consistent and has no free variables.

c) False. We cannot tell anything about the system just knowing this information, as it might have no solutions, or exactly one solution, or infinitely many solutions. For example,

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

2. Is the matrix below in reduced row echelon form?

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & -3 & 1 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Solution.

Yes.

3. Put an augmented matrix into reduced row echelon form to solve the system

$$x_1 - 2x_2 - 9x_3 + x_4 = 3$$
$$4x_2 + 8x_3 - 24x_4 = 4.$$

Solution.

$$\left(\begin{array}{cccc|c} 1 & -2 & -9 & 1 & 3 \\ 0 & 4 & 8 & -24 & 4 \end{array}\right) \xrightarrow{R_2=\frac{R_2}{4}} \left(\begin{array}{cccc|c} 1 & -2 & -9 & 1 & 3 \\ 0 & 1 & 2 & -6 & 1 \end{array}\right) \xrightarrow{R_1=R_1+2R_2} \left(\begin{array}{cccc|c} \boxed{1} & 0 & -5 & -11 & 5 \\ 0 & \boxed{1} & 2 & -6 & 1 \end{array}\right)$$

The third and fourth columns are not pivot columns, so x_3 and x_4 are free variables. Our equations are

$$\begin{aligned} x_1 - 5x_3 - 11x_4 &= 5 \\ x_2 + 2x_3 - 6x_4 &= 1. \end{aligned}$$

Therefore,

$$\begin{aligned} x_1 &= 5 + 5x_3 + 11x_4 \\ x_2 &= 1 - 2x_3 + 6x_4 \\ x_3 &= x_3 \qquad \text{(any real number)} \\ x_4 &= x_4 \qquad \text{(any real number)} \end{aligned}$$

4. **a)** Row reduce the following matrices to reduced row echelon form.

b) If these are augmented matrices for a linear system (with the last column being after the $=$ sign), then which are inconsistent? Which have a *unique* solution?

Solution.

$$\begin{aligned}
\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{pmatrix}
&\xrightarrow{R_2 = R_2 - 4R_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 6 & 7 & 8 & 9 \end{pmatrix} \\
&\xrightarrow{R_3 = R_3 - 6R_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{pmatrix} \\
&\xrightarrow{R_2 = R_2 \div -3} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -5 & -10 & -15 \end{pmatrix} \\
&\xrightarrow{R_3 = R_3 + 5R_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&\xrightarrow{R_1 = R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

This is the reduced row echelon form. Interpreted as an augmented matrix, it corresponds to the system of linear equations

$$\begin{aligned} x \qquad - z &= -2 \\ y + 2z &= 3 \\ 0 &= 0. \end{aligned}$$

This system is consistent, but since z is a free variable, it does not have a *unique* solution.

$$\begin{pmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{pmatrix} \xrightarrow{R_2 = R_2 - 3R_1} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 5 & 7 & 9 & 1 \end{pmatrix}$$

$$\xrightarrow{R_3 = R_3 - 5R_1} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{pmatrix}$$

$$\xrightarrow{R_2 = R_2 \div -4} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & -8 & -16 & -34 \end{pmatrix}$$

$$\xrightarrow{R_3 = R_3 + 8R_2} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -10 \end{pmatrix}$$

$$\xrightarrow{R_3 = R_3 \div -10} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1 = R_1 - 7R_3} \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2 = R_2 - 3R_3} \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1 = R_1 - 3R_2} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is the reduced row echelon form. Interpreted as an augmented matrix, it corresponds to the system of linear equations

$$\begin{aligned} x \quad - z &= 0 \\ y + 2z &= 0 \\ 0 &= 1, \end{aligned}$$

which is inconsistent.

$$\begin{pmatrix} 3 & -4 & 2 & 0 \\ -8 & 12 & -4 & 0 \\ -6 & 8 & -1 & 0 \end{pmatrix} \xrightarrow{R_2 = R_2 + 3R_1} \begin{pmatrix} 3 & -4 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ -6 & 8 & -1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 \longleftrightarrow R_2} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 3 & -4 & 2 & 0 \\ -6 & 8 & -1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 = R_2 - 3R_1} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & -4 & -4 & 0 \\ -6 & 8 & -1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3 = R_3 + 6R_1} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & 8 & 11 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 = R_2 \div -4} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 8 & 11 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3 = R_3 - 8R_2} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3 = R_3 \div 3} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 = R_1 - 2R_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 = R_2 - R_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

This is the reduced row echelon form. Interpreted as an augmented matrix, it corresponds to the system of linear equations

$$x = 0 \quad y = 0 \quad z = 0,$$

which has a unique solution.

5. We can use linear algebra to find a polynomial that fits given data, in the same way that we found a circle through three specified points in the Webwork.

Is there a degree-three polynomial $P(x)$ whose graph passes through the points $(-2,6)$, $(-1,4)$, $(1,6)$, and $(2,22)$? If so, how many degree-three polynomials have a graph through those four points? We answer this question in steps below.

a) If $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ is a degree-three polynomial passing through the four points listed above, then $P(-2) = 6$, $P(-1) = 4$, $P(1) = 6$, and $P(2) = 22$. Write a system of four equations which we would solve to find a_0, a_1, a_2, and a_3.

b) Write the augmented matrix to represent this system and put it into reduced row-echelon form. Is the system consistent? How many solutions does it have?

Solution.

a) We compute

$$\begin{aligned}
P(-2) = 6 \quad &\implies \quad a_0 + a_1 \cdot (-2) + a_2 \cdot (-2)^2 + a_3 \cdot (-2)^3 = 6, \\
P(-1) = 4 \quad &\implies \quad a_0 + a_1 \cdot (-1) + a_2 \cdot (-1)^2 + a_3 \cdot (-1)^3 = 4, \\
P(1) = 6 \quad &\implies \quad a_0 + a_1 \cdot 1 + a_2 \cdot 1^2 + a_3 \cdot 1^3 = 6, \\
P(2) = 22 \quad &\implies \quad a_0 + a_1 \cdot 2 + a_2 \cdot 2^2 + a_3 \cdot 2^3 = 22.
\end{aligned}$$

Simplifying gives us

$$\begin{aligned} a_0 - 2a_1 + 4a_2 - 8a_3 &= 6 \\ a_0 - a_1 + a_2 - a_3 &= 4 \\ a_0 + a_1 + a_2 + a_3 &= 6 \\ a_0 + 2a_1 + 4a_2 + 8a_3 &= 22. \end{aligned}$$

b) The corresponding augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & -2 & 4 & -8 & 6 \\ 1 & -1 & 1 & -1 & 4 \\ 1 & 1 & 1 & 1 & 6 \\ 1 & 2 & 4 & 8 & 22 \end{array}\right)$$

We label pivots with boxes as we proceed along. First, we subtract row 1 from each of rows 2, 3, and 4.

$$\left(\begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6 \\ 1 & -1 & 1 & -1 & 4 \\ 1 & 1 & 1 & 1 & 6 \\ 1 & 2 & 4 & 8 & 22 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6 \\ 0 & \boxed{1} & -3 & 7 & -2 \\ 0 & 3 & -3 & 9 & 0 \\ 0 & 4 & 0 & 16 & 16 \end{array}\right)$$

We now create zeros below the second pivot by subtracting multiples of the second row, then divide by 6 to get

$$\left(\begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6 \\ 0 & \boxed{1} & -3 & 7 & -2 \\ 0 & 0 & \boxed{6} & -12 & 6 \\ 0 & 0 & 12 & -12 & 24 \end{array}\right) \xrightarrow[\rightsquigarrow]{R_3 = R_3 \div 6} \left(\begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6 \\ 0 & \boxed{1} & -3 & 7 & -2 \\ 0 & 0 & \boxed{1} & -2 & 1 \\ 0 & 0 & 12 & -12 & 24 \end{array}\right).$$

Now we subtract a 12 times row 3 from row 4 and divide by 12:

$$\left(\begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6 \\ 0 & \boxed{1} & -3 & 7 & -2 \\ 0 & 0 & \boxed{1} & -2 & 1 \\ 0 & 0 & 0 & \boxed{12} & 12 \end{array}\right) \xrightarrow[\rightsquigarrow]{R_4 = R_4 \div 12} \left(\begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6 \\ 0 & \boxed{1} & -3 & 7 & -2 \\ 0 & 0 & \boxed{1} & -2 & 1 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{array}\right).$$

At this point we can actually use back-substitution to solve: the last row says $a_3 = 1$, then plugging in $a_3 = 1$ in the third row gives us $a_2 = 3$, etc. However, for the sake of practice with reduced echelon form, let's keep row-reducing.

From right to left, we create zeros above the pivots in pivot columns by subtracting multiples of the pivot columns.

$$\left(\begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6 \\ 0 & \boxed{1} & -3 & 7 & -2 \\ 0 & 0 & \boxed{1} & -2 & 1 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} \boxed{1} & -2 & 4 & 0 & 14 \\ 0 & \boxed{1} & -3 & 0 & -9 \\ 0 & 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|c} \boxed{1} & -2 & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{array}\right)$$

So $a_0 = 2$, $a_1 = 0$, $a_2 = 3$, and $a_3 = 1$. In other words,

$$\boxed{P(x) = 2 + 3x^2 + x^3}.$$

Therefore, there is exactly one third-degree polynomial satisfying the conditions of the problem. (You should check that, in fact, we have $P(-2) = 6$, $P(-1) = 4$, etc.)

6. Consider the linear equation in the variables x_1, x_2, and x_3 given by

$$x_1 - x_2 + x_3 = 5.$$

If we write the general solution to this system in parametric form, we will find that x_2 and x_3 are free and so

$$x_1 = x_2 - x_3 + 5, \qquad x_2 = x_2 \quad (x_2 \text{ real}), \qquad x_3 = x_3 \quad (x_3 \text{ real}).$$

Is the following also a parametrization of the solution set?

$$x_1 = x_1 \quad (x_1 \text{ real}), \qquad x_2 = x_2 \quad (x_2 \text{ real}), \qquad x_3 = -x_1 + x_2 + 5 \quad (x_3 \text{ real}),$$

Solution.

Yes. The second parametrization might look different than the first, but it still describes the same set of points: if we expand $x_1 - x_2 + x_3$, we obtain

$$x_1 - x_2 + x_3 = x_1 - x_2 + (-x_1 + x_2 + 5) = 5.$$

The difference is that due to the way we assign free variables in parametric form, here we get x_2 and x_3 free so that x_1 depends on x_2 and x_3, whereas the second parametrization treats x_1 and x_2 as free so that x_3 depends on x_1 and x_2. In both cases, the parametrizations describe the plane in $\mathbf{R}^3$ given by $x_1 - x_2 + x_3 = 5$.

By designating one standard “parametric form" and having everyone write solution sets in that form, we avoid having to go through the extra step of trying to determine whether one parametrization is equivalent to another.