MATH 1553, SPRING 2022
MIDTERM 1

Name	

Circle your lecture below.

Jankowski, lecture A (8:25-9:15 AM) Jankowski, lecture D (9:30-10:20 AM)

Yu, lecture G (12:30-1:20 PM)

Leykin, lecture I (2:00-2:50 PM) Leykin, lecture M (3:30-4:20 PM)

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 70 points, and you have 75 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means “reduced row echelon form."
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and all work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

Please read and sign the following statement.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 9:15 PM on Wednesday, February 9.

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Problem 1.

True or false. If the statement is *ever* false, circle FALSE. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

a) If A is a 5×4 matrix A with a pivot in every column, then every vector in $\mathbf{R}^5$ must be in the span of the columns of A.

TRUE FALSE

b) Suppose we are given a consistent system of 4 linear equations in 3 variables and the corresponding augmented matrix has 2 pivots in its reduced row echelon form. Then the set of solutions to the system of linear equations is a plane.

TRUE FALSE

c) The following vector equation is consistent for each b in $\mathbf{R}^2$:

$$x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = b.$$

TRUE FALSE

d) If v_1, v_2, and v_3 are vectors in $\mathbf{R}^2$, then we must have $\text{Span}\{v_1, v_2, v_3\} = \mathbf{R}^2$.

TRUE FALSE

e) If A is a 2×3 matrix and b is a vector so that the set of solutions to $Ax = b$ is the span of $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, then $b = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

TRUE FALSE

Solution.

a) False by a standard pivot counting argument. In order for every b in $\mathbf{R}^5$ to be in the column span of A, we need A to have a pivot in every **row** (i.e. 5 pivots), but A has exactly 4 pivots.

b) False by a standard pivot counting argument. The augmented matrix has 2 pivots to the left of the augment bar and 3 columns left of the augment bar, so the system will have exactly 1 free variable and therefore the solution set will be the line.

c) True. The span of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is clearly all of $\mathbf{R}^2$ (one way to see this is that the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ has a pivot in each row), which is precisely to say that the given vector equation is consistent for every b in $\mathbf{R}^2$.

d) False. Just take $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, and $v_3 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$. Then $\text{Span}\{v_1, v_2, v_3\}$ is only the x_1-axis. Inspired by #2 on the 2.1-2.2 Worksheet, and also similar to #7 on the 2.4 Supplement.

e) True. The span of $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ includes the zero vector. Therefore, $x = 0$ is a solution, so $b = A(0) = 0$.

Problem 2.

Multiple choice and short answer. You do not need to show work or justify your answers.

a) (3 pts) Answer YES or NO to each of the following questions.

(i) Is the equation $x + \ln(2)y = 4$ a linear equation in x and y? $\boxed{\text{YES}}$ NO

(ii) Is $(2,0,1)$ in the plane $x - y - z = 1$? $\boxed{\text{YES}}$ NO

(iii) Is it possible for a system of linear equations to have exactly 3 solutions?
YES $\boxed{\text{NO}}$

b) (2 pts) Write an *inconsistent* system of two linear equations in the three variables x, y, and z.

c) (3 pts) Which of the following matrices are in reduced row echelon form? Circle all that apply.

(i) $\left(\begin{array}{ccc|c} 0 & 0 & 1 & 1 \end{array}\right)$ YES

(ii) $\left(\begin{array}{ccc|c} 1 & -5 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$ YES

(iii) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$ YES

d) (2 pts) Suppose an augmented matrix in RREF has $\left(\begin{array}{ccc|c} 0 & 0 & 1 & 3 \end{array}\right)$ as its bottom row. Which one of the following statements is true about the system of linear equations corresponding to the augmented matrix?

(i) The system must be inconsistent.

(ii) We need more information to determine whether the system is consistent or inconsistent.

(iii) The system must have exactly one solution.

(iv) The system must have infinitely many solutions.

(v) The system must be consistent, but we need more information to determine whether it has a unique solution or infinitely many solutions. CORRECT

Solution.

a) (i) Yes, it is linear. Taken from 1.1 Webwork.
(ii) Yes, because $2 - 0 - 1 = 1$.
(iii) No, a consistent linear system has either exactly one solution or infinitely many solutions.

b) Basically taken from the 1.1 Webwork. Many examples possible, for example

$$x + y + z = 1,$$
$$x + y + z = 2.$$

c) All three are in RREF.

d) (v). The system must be consistent because its rightmost pivot is left of the vertical bar, but we need more information to determine the number of solutions. We give one example of each below.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right), \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right).$$

Problem 3.

Multiple choice and short answer. You do not need to show your work or justify your answers.

a) (3 pts) Write a set of three different vectors v_1, v_2, and v_3 in $\mathbf{R}^3$ satisfying both of the following properties:

- The span of any two of the vectors is a plane.
- Span$\{v_1, v_2, v_3\}$ is also a plane.

b) (2 pts) You are trying to solve this system of equations, but there is a smudge on the page on the coefficient of y in the first equation (the smudge is the mark $\bullet$ below):

$$\begin{aligned} 2x + \bullet y &= 5 \\ 4x - y &= 3 \end{aligned}$$

The answer key says that the solution to the system is $(1,1)$. Which one of the following scenarios is possible?

(i) There must have been a typo, since $(1,1)$ cannot be a solution.

(ii) There is exactly one value for the smudge so that $(1,1)$ is a solution.

(iii) There are multiple possible values for the smudge so that $(1,1)$ is a solution.

c) (5 pts) Suppose A is a matrix and b is a nonzero vector so that the solution set to $Ax = b$ has the following parametric vector form (where x_2 and x_3 are free):

$$\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

(i) How many columns does A have? Circle your answer below.

2   3   4   not enough information given

(ii) How many rows does A have? Circle your answer below.

2   3   4   not enough information given

(iii) Which of the following vectors must satisfy the homogeneous equation $Ax = 0$? Circle all that apply.

(1) $x = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ (2) $x = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ (3) $x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Solution.

a) Taken directly from the 2.1-2.2 Webwork. Many examples are possible, for example

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

b) The correct answer is (ii): $(1,1)$ satisfies $4x - y = 3$, and in order to satisfy $2(1) + \bullet(1) = 5$, we need the smudge to be 3.

c) (i) 3 columns, since the solutions are vectors in $\mathbf{R}^3$.
(ii) Not enough info. We aren't given any information about the number of rows of A or the number of entries of b. For example,

$$A = \begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

satisfies the conditions given, but so does

$$A = \begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ 1 & 1 & -2 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

. (iii) The vectors (2) and (3) satisfy $Ax = 0$.
Note that the vectors associated to x_2 and x_3 in the parametric vector form satisfy $Ax = 0$. Here, $x = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ is associated to the x_2 term in parametric vector form, while $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is the sum of $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$.

Note that $A\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = b$, which is a nonzero vector.

Problem 4.

Short answer and multiple choice. Show your work on part (a).

a) (2 pts) Compute the matrix product $\begin{pmatrix} 1 & 0 & 3 \\ -2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$.

b) (2 pts) Suppose A is a matrix and $A\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$. Compute $A\begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

c) (3 pts) For each augmented matrix given below, determine whether the corresponding linear system is consistent. If it is inconsistent, circle "there are no solutions." If the system is consistent, determine whether the solution set is a point, a line, or a plane, and circle your answer.

(i) For $\left(\begin{array}{cc|c} 1 & 0 & 2021 \\ 0 & 0 & 2022 \end{array}\right)$, the solution set is:
a single point   a line   a plane   there are no solutions

(ii) For $\left(\begin{array}{cc|c} 1 & 2 & 1011 \\ 2 & 4 & 2022 \end{array}\right)$, the solution set is:
a single point   a line   a plane   there are no solutions

(iii) For $\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$, the solution set is:
a single point   a line   a plane   there are no solutions

d) (3 pts) Suppose v_1, v_2, v_3, and b are vectors in $\mathbf{R}^n$. Which of the following statements are true? Circle all that apply.

(i) If b is in Span$\{v_1, v_2, v_3\}$, then the vector equation $x_1v_1 + x_2v_2 + x_3v_3 = b$ must be consistent.

(ii) If the vector equation $x_1v_1 + x_2v_2 = b$ is consistent, then the vector equation

$$x_1v_1 + x_2v_2 + x_3v_3 = b$$

must also be consistent.

(iii) If b is the zero vector, then the vector equation $x_1v_1 + x_2v_2 + x_3v_3 = b$ must be consistent.

Solution.

a) Standard computation similar to 2.3 Webwork #1.

$$\begin{pmatrix} 1 & 0 & 3 \\ -2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = 0\begin{pmatrix} 1 \\ -2 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 3 \end{pmatrix} + 3\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 9 \\ 3 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}.$$

b) Standard computation similar to 2.3 Webwork #2: $\begin{pmatrix}4\\2\end{pmatrix} = 2\begin{pmatrix}2\\1\end{pmatrix}$, so by linearity of matrix multiplication of vectors:

$$A\begin{pmatrix}4\\2\end{pmatrix} = A\left(2\begin{pmatrix}2\\1\end{pmatrix}\right) = 2A\begin{pmatrix}2\\1\end{pmatrix} = 2\begin{pmatrix}6\\-2\end{pmatrix} = \begin{pmatrix}12\\-4\end{pmatrix}.$$

c) Straightforward with chapter 1 material.
(i) There are no solutions.
(ii) The solution set is a line: the matrix row reduces to $\left(\begin{array}{cc|c}1 & 2 & 1011\\0 & 0 & 0\end{array}\right)$, so the system is consistent and there is one variable.
(iii) There are two free variables, so the solution set is a plane. This is the linear system of equations given by

$$\begin{aligned}0x_1 + 0x_2 &= 0\\0x_1 + 0x_2 &= 0,\end{aligned}$$

which is satisfied by every $\begin{pmatrix}x_1\\x_2\end{pmatrix}$ in $\mathbf{R}^2$.

d) (i), (ii), and (iii) are all true. Part (i) is a direct consequence of the definition of span, part (ii) is a consequence of the definition of span, and (iii) is a standard fact that we have emphasized.

For (i), if b is in Span$\{v_1, v_2, v_3\}$ then b is a linear combination of v_1, v_2, v_3, which is precisely to say that $x_1v_1 + x_2v_2 + x_3v_3 = b$ is consistent.

For (ii), note that if $x_1v_1 + x_2v_2 = b$ is consistent then b is already in Span$\{v_1, v_2\}$ which is contained in Span$\{v_1, v_2, v_3\}$, so b is in Span$\{v_1, v_2, v_3\}$. Alternatively, we could see this directly: if $c_1v_1 + c_2v_2 = b$ for some scalars c_1 and c_2 then $c_1v_1 + c_2v_2 + 0v_3 = b$.

For (iii), we have seen it repeatedly emphasized that we can always write the zero vector as a linear combination of any number of vectors:

$$0v_1 + 0v_2 + 0v_3 = 0.$$

Problem 5.

Free response. Show your work! The two parts of this problem are unrelated.

a) (5 pts) Consider the linear system of equations given by

$$2x - hy = 4$$
$$8x + 16y = k.$$

Find all values of h and k (if there are any) so that the system has infinitely many solutions.

b) (5 pts) Find all values of c so that $\begin{pmatrix} 5 \\ c \\ -2 \end{pmatrix}$ is a linear combination of $\begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

Solution.

a) This is an easier version of #4a from the sample midterm. We row-reduce an augmented matrix:

$$\left(\begin{array}{cc|c} 2 & -h & 4 \\ 8 & 16 & k \end{array}\right) \xrightarrow{R_2 = R_2 - 4R_1} \left(\begin{array}{cc|c} 2 & -h & 4 \\ 0 & 16+4h & k-16 \end{array}\right).$$

For the corresponding system to have infinitely many solutions, we need $16+4h=0$ so that we will have a free variable, and we also need $k-16=0$ (so that the system is consistent!). Therefore,

$$\boxed{h=-4, \qquad k=16}.$$

b) This is almost identical to #4b from the sample midterm. We row reduce:

$$\left(\begin{array}{cc|c} 1 & 2 & 5 \\ 4 & 1 & c \\ 5 & 1 & -2 \end{array}\right) \xrightarrow[R_3 = R_3 - 5R_1]{R_2 = R_2 - 4R_1} \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -7 & c-20 \\ 0 & -9 & -27 \end{array}\right) \xrightarrow[\text{then } R_2 = R_2/-9]{R_2 \leftrightarrow R_3} \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & -7 & c-20 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & c+1 \end{array}\right)$$

This system will be consistent when there is no pivot in the rightmost column, so we precisely need $c+1=0$, thus $\boxed{c=-1}$.

On this problem, some students did not finish row-reducing the matrix into an REF and instead made the serious error of concluding $c=20$ or that $-7=c-20$.

Problem 6.

Free response. Show your work on parts (a) and (b).

Consider the following linear system of equations in the variables $x_1,\ x_2,\ x_3,\ x_4$:

$$x_1 - 4x_2 + 0x_3 + 2x_4 = 1$$

$$-x_1 + 4x_2 - x_3 - x_4 = -3$$

$$-2x_1 + 8x_2 - 2x_3 - 2x_4 = -6.$$

a) (4 pts) Write the augmented matrix corresponding to this system, and put the augmented matrix into RREF.

b) (4 pts) The system is consistent. Write the set of solutions to the system of equations in parametric vector form.

c) (2 pts) Write *one* vector which is not the zero vector and which is a solution to the corresponding homogeneous system of equations given below:

$$x_1 - 4x_2 + 0x_3 + 2x_4 = 0$$

$$-x_1 + 4x_2 - x_3 - x_4 = 0$$

$$-2x_1 + 8x_2 - 2x_3 - 2x_4 = 0.$$

There is no work necessary and no partial credit on part (c).

Solution.

This problem is a slight modification of #5 from the sample exam.

a) We row-reduce:

$$\left(\begin{array}{cccc|c} 1 & -4 & 0 & 2 & 1 \\ -1 & 4 & -1 & -1 & -3 \\ -2 & 8 & -2 & -2 & -6 \end{array}\right) \xrightarrow[R_3=R_3+2R_1]{R_2=R_2+R_1} \left(\begin{array}{cccc|c} 1 & -4 & 0 & 2 & 1 \\ 0 & 0 & -1 & 1 & -2 \\ 0 & 0 & -2 & 2 & -4 \end{array}\right) \xrightarrow[\text{then } R_2=-R_2]{R_3=R_3-R_2} \left(\begin{array}{cccc|c} \boxed{1} & -4 & 0 & 2 & 1 \\ 0 & 0 & \boxed{1} & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Students were not required to label the pivots, but we did for the sake of clarity.

b) From the augmented matrix in RREF in (a), we see x_2 and x_4 are free and

$$x_1 = 1 + 4x_2 - 2x_4, \qquad x_2 = x_2 \ \ (x_2 \text{ real}), \qquad x_3 = 2 + x_4, \qquad x_4 = x_4 \ \ (x_4 \text{ real}).$$

Thus

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 + 4x_2 - 2x_4 \\ x_2 \\ 2 + x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

c) The vectors $\begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ satisfy the corresponding homogeneous system, and so will any linear combination of these two vectors.

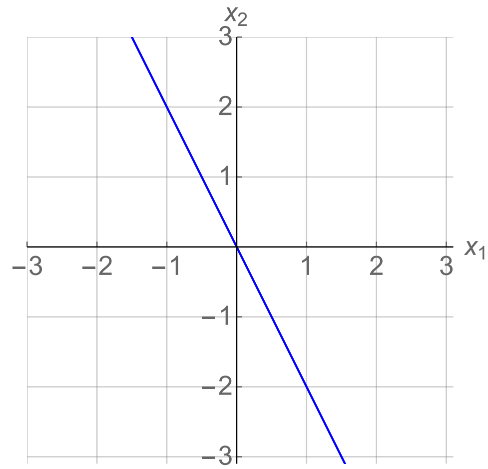
Note that $\begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$ is not a correct answer here. It satisfies the **original** system of equations (with 1, -3 and -6 to the right of the equals signs).

Problem 7.

Show your work on parts (a) and (b).

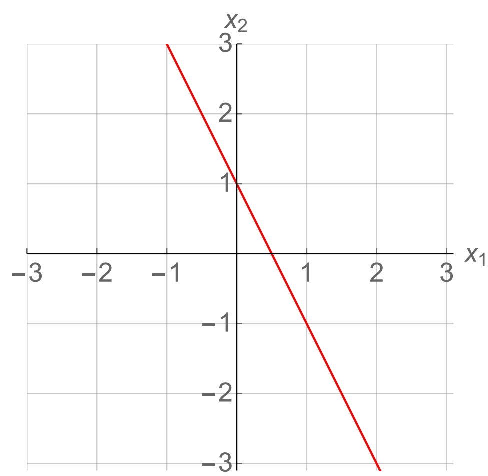
For this entire problem, let $A = \begin{pmatrix} 2 & 1 \\ -8 & -4 \\ 4 & 2 \end{pmatrix}$.

a) (5 pts) Write the solution set to $Ax = 0$ in parametric form, and draw the solution set to $Ax = 0$ on the graph below.



b) (3 pts) For some b, the vector $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a solution to the equation $Ax = b$.

Do we have enough information to draw the solution set for $Ax = b$? If your answer is no, explain why we do not have enough information. If your answer is yes, draw the solution set to $Ax = b$ on the graph below.



c) (2 pts) You do not need to show work on this part. Answer the following.
The span of the columns of A is a:

(circle one answer) **point** **line** **plane**

in:

(circle one answer) $\mathbf{R}$ $\mathbf{R}^2$ $\mathbf{R}^3$.

Solution.

a) $\left(\begin{array}{cc|c} 2 & 1 & 0 \\ -8 & -4 & 0 \\ 4 & 2 & 0 \end{array}\right) \xrightarrow[R_3=R_3-2R_1]{R_2=R_2+4R_1} \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$. Therefore the solution set is

$$x_1 = -\frac{x_2}{2}, \quad x_2 = x_2 \ \ (x_2 \text{ real}).$$

It does not matter in what way the student indicates that x_2 is free, as long as it is made clear in some fashion that x_2 is free (either by saying it is free, or writing $x_2 = x_2$ (x_2 real), or something similar).

The graph is just the span of $\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$.

b) We know the solution set to $Ax = 0$, and we know that $Av = b$, so the solution set to $Ax = b$ is the line through v parallel to the solution set to $Ax = 0$.

c) The two columns of A are nonzero scalar multiples of one another, so their span is a line. Also, each column of A is a vector in $\mathbf{R}^3$. Therefore, the column span of A a line in $\mathbf{R}^3$.

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.