

Eigenvectors and Eigenvalues

Reminder

Definition

Let A be an $n \times n$ matrix.

1. An **eigenvector** of A is a nonzero vector v in \mathbf{R}^n such that $Av = \lambda v$, for some λ in \mathbf{R} .
2. An **eigenvalue** of A is a number λ in \mathbf{R} such that the equation $Av = \lambda v$ has a nontrivial solution.
3. If λ is an eigenvalue of A , the λ -**eigenspace** is the solution set of $(A - \lambda I_n)x = 0$.

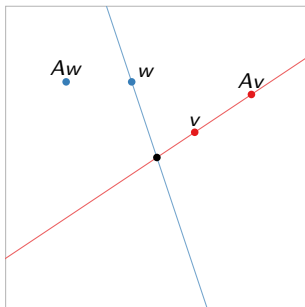
Eigenspaces

Geometry

Eigenvectors, geometrically

An eigenvector of a matrix A is a nonzero vector v such that:

- ▶ Av is a multiple of v , which means
- ▶ Av is collinear with v , which means
- ▶ Av and v are *on the same line through the origin*.



v is an eigenvector

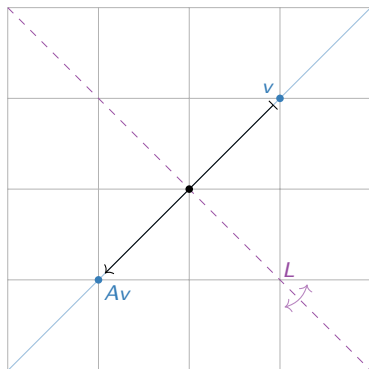
w is not an eigenvector

Eigenspaces

Geometry; example

Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T .

Question: What are the eigenvalues and eigenspaces of A ? No computations!



Does anyone see any eigenvectors
(vectors that don't move off their line)?

v is an eigenvector with eigenvalue -1 .

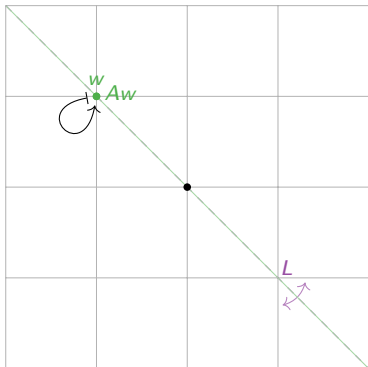
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Eigenspaces

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w is an eigenvector with eigenvalue 1.

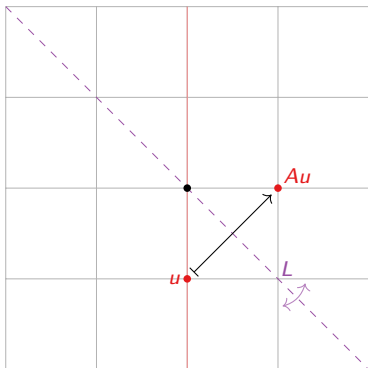
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Eigenspaces

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u is *not* an eigenvector.

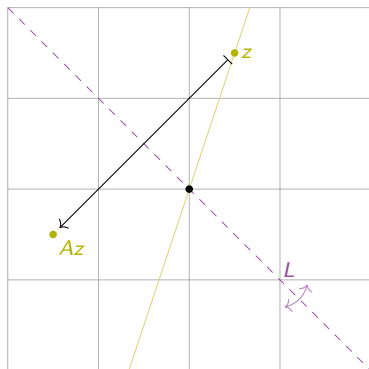
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Eigenspaces

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Neither is z .

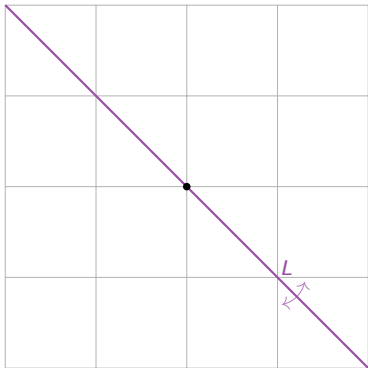
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The 1-eigenspace is L
(all the vectors x where $Ax = x$).

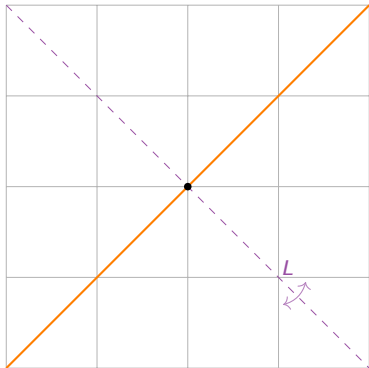
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Eigenspaces

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The (-1) -eigenspace is **the line $y = x$**
(all the vectors x where $Ax = -x$).

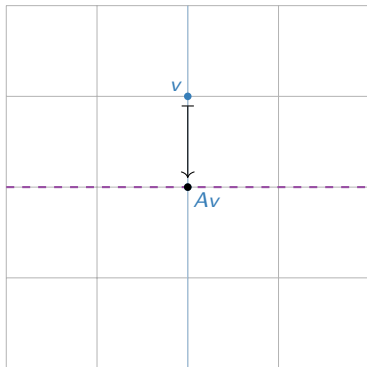
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Eigenspaces

Geometry; example

Let $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be the vertical projection onto the x -axis, and let A be the matrix for T .

Question: What are the eigenvalues and eigenspaces of A ? No computations!



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v is an eigenvector with eigenvalue 0.

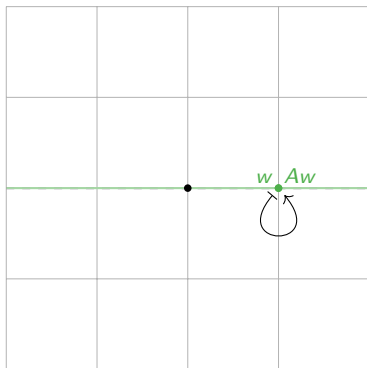
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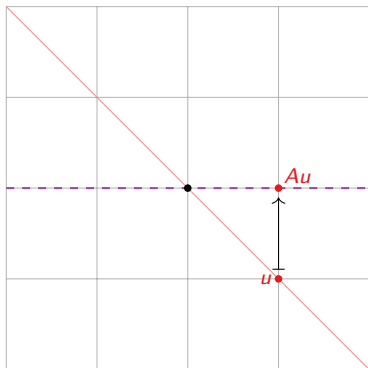
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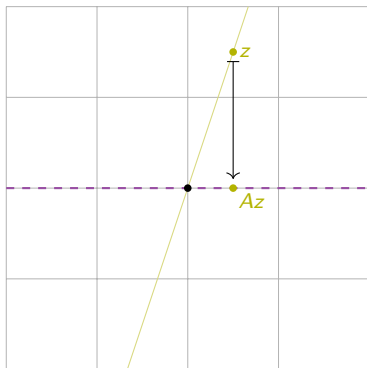
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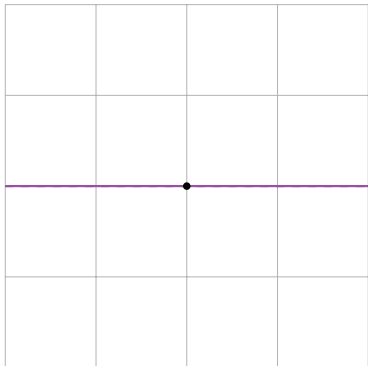
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The 1-eigenspace is **the x -axis** (all the vectors x where $Ax = x$).

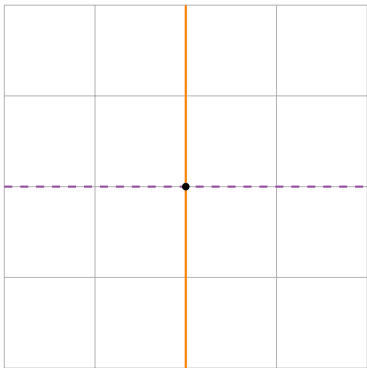
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The 0-eigenspace is **the y -axis**
(all the vectors x where $Ax = 0x$).

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Eigenspaces

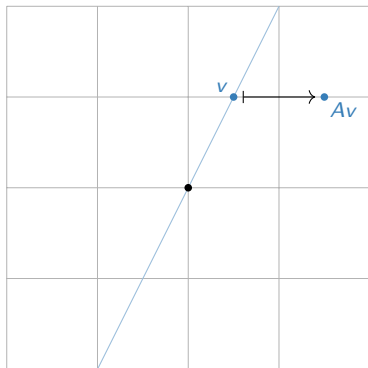
Geometry; example

Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

so $T(x) = Ax$ is a shear in the x -direction.

Question: What are the eigenvalues and eigenspaces of A ? No computations!



Does anyone see any eigenvectors (vectors that don't move off their line)?

Vectors v above the x -axis are moved right but not up...
so they're not eigenvectors.

[interactive]

Eigenspaces

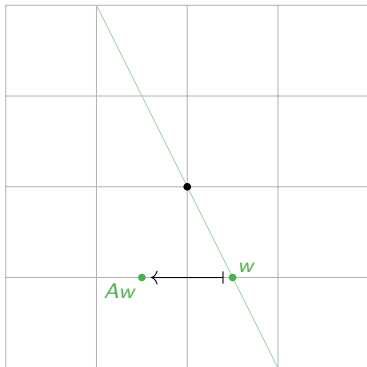
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[interactive]

Eigenspaces

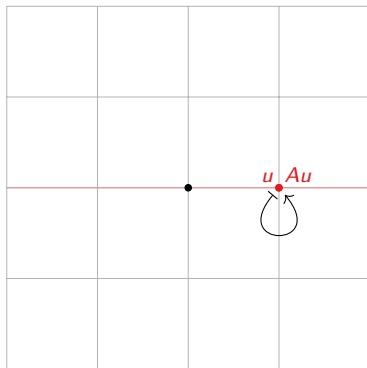
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[interactive]

Eigenspaces

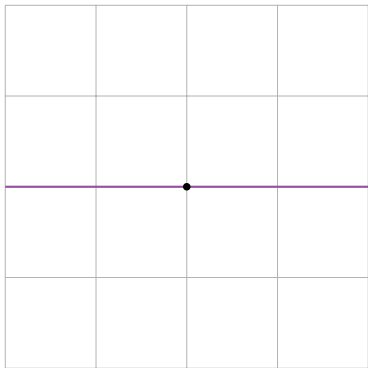
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[interactive]

Eigenspaces

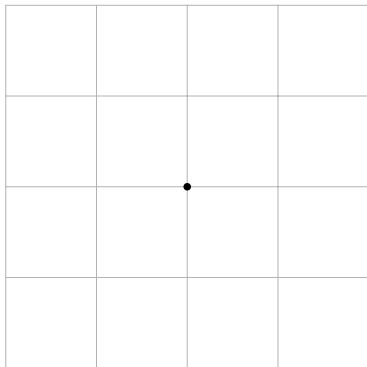
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Question: What are the eigenvalues and eigenspaces of A ? No computations!

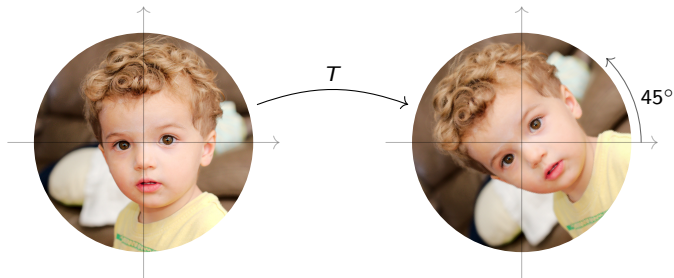


Does anyone see any eigenvectors
(vectors that don't move off their line)?

There are no other eigenvectors.

[interactive]

Let $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be counterclockwise rotation by 45° , and let A be the matrix for T .



Poll

Find an eigenvector of A without doing any computations.

- A.** Okay. **B.** No way.

Answer: **B.** No way. There are no eigenvectors!

[interactive]

Section 5.2

The Characteristic Polynomial

The Characteristic Polynomial

Let A be a square matrix.

$$\begin{aligned}\lambda \text{ is an eigenvalue of } A &\iff Ax = \lambda x \text{ has a nontrivial solution} \\ &\iff (A - \lambda I)x = 0 \text{ has a nontrivial solution} \\ &\iff A - \lambda I \text{ is not invertible} \\ &\iff \det(A - \lambda I) = 0.\end{aligned}$$

This gives us a way to compute the eigenvalues of A .

Definition

Let A be a square matrix. The **characteristic polynomial** of A is

$$f(\lambda) = \det(A - \lambda I).$$

The **characteristic equation** of A is the equation

$$f(\lambda) = \det(A - \lambda I) = 0.$$

Important

The eigenvalues of A are the roots of the characteristic polynomial $f(\lambda) = \det(A - \lambda I)$.

The Characteristic Polynomial

Example

Question: What are the eigenvalues of

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}?$$

Answer: First we find the characteristic polynomial:

$$\begin{aligned} f(\lambda) &= \det(A - \lambda I) = \det \left[\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right] = \det \begin{pmatrix} 5 - \lambda & 2 \\ 2 & 1 - \lambda \end{pmatrix} \\ &= (5 - \lambda)(1 - \lambda) - 2 \cdot 2 \\ &= \lambda^2 - 6\lambda + 1. \end{aligned}$$

The eigenvalues are the roots of the characteristic polynomial, which we can find using the quadratic formula:

$$\lambda = \frac{6 \pm \sqrt{36 - 4}}{2} = 3 \pm 2\sqrt{2}.$$

The Characteristic Polynomial

Example

Question: What is the characteristic polynomial of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}?$$

Answer:

$$\begin{aligned} f(\lambda) &= \det(A - \lambda I) = \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = (a - \lambda)(d - \lambda) - bc \\ &= \lambda^2 - (a + d)\lambda + (ad - bc) \end{aligned}$$

What do you notice about $f(\lambda)$?

- ▶ The constant term is $\det(A)$, which is zero if and only if $\lambda = 0$ is a root.
- ▶ The linear term $-(a + d)$ is the negative of the sum of the diagonal entries of A .

Definition

The **trace** of a square matrix A is $\text{Tr}(A) =$ sum of the diagonal entries of A .

Shortcut

The characteristic polynomial of a 2×2 matrix A is

$$f(\lambda) = \lambda^2 - \text{Tr}(A)\lambda + \det(A).$$

The Characteristic Polynomial

Example

Question: What are the eigenvalues of the rabbit population matrix

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}?$$

Answer: First we find the characteristic polynomial:

$$\begin{aligned} f(\lambda) &= \det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 6 & 8 \\ \frac{1}{2} & -\lambda & 0 \\ 0 & \frac{1}{2} & -\lambda \end{pmatrix} \\ &= 8 \left(\frac{1}{4} - 0 \cdot -\lambda \right) - \lambda \left(\lambda^2 - 6 \cdot \frac{1}{2} \right) \\ &= -\lambda^3 + 3\lambda + 2. \end{aligned}$$

We know from before that one eigenvalue is $\lambda = 2$: indeed, $f(2) = -8 + 6 + 2 = 0$. Doing polynomial long division, we get:

$$\frac{-\lambda^3 + 3\lambda + 2}{\lambda - 2} = -\lambda^2 - 2\lambda - 1 = -(\lambda + 1)^2.$$

Hence $\lambda = -1$ is also an eigenvalue.

Factoring the Characteristic Polynomial

It's easy to factor quadratic polynomials:

$$x^2 + bx + c = 0 \implies x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

It's less easy to factor cubics, quartics, and so on:

$$x^3 + bx^2 + cx + d = 0 \implies x = ???$$

$$x^4 + bx^3 + cx^2 + dx + e = 0 \implies x = ???$$

Read about factoring polynomials by hand in §5.2.

We did two different things today.

First we talked about the geometry of eigenvalues and eigenvectors:

- ▶ Eigenvectors are vectors v such that v and Av are on the same line through the origin.
- ▶ You can pick out the eigenvectors geometrically if you have a picture of the associated transformation.

Then we talked about characteristic polynomials:

- ▶ We learned to find the eigenvalues of a matrix by computing the roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$.
- ▶ For a 2×2 matrix A , the characteristic polynomial is just

$$p(\lambda) = \lambda^2 - \text{Tr}(A)\lambda + \det(A).$$