Math 1553 Worksheet §§3.5-4.1

Solutions

1. True or false (justify your answer). Answer true if the statement is *always* true. Otherwise, answer false.

a) If A and B are $n \times n$ matrices and both are invertible, then the inverse of AB is $A^{-1}B^{-1}$.

b) If A is an $n \times n$ matrix and the equation $Ax = b$ has at least one solution for each b in $\mathbf{R}^n$, then the solution is *unique* for each b in $\mathbf{R}^n$.

c) If A is an $n \times n$ matrix and the equation $Ax = b$ has at most one solution for each b in $\mathbf{R}^n$, then the solution must be *unique* for each b in $\mathbf{R}^n$.

d) If A and B are invertible $n \times n$ matrices, then $A+B$ is invertible and $(A+B)^{-1} = A^{-1} + B^{-1}$.

e) If A and B are $n \times n$ matrices and $ABx = 0$ has a unique solution, then $Ax = 0$ has a unique solution.

f) If A is a 3×4 matrix and B is a 4×2 matrix, then the linear transformation Z defined by $Z(x) = ABx$ has domain $\mathbf{R}^3$ and codomain $\mathbf{R}^2$.

g) Suppose A is an $n \times n$ matrix and every vector in $\mathbf{R}^n$ can be written as a linear combination of the columns of A. Then A must be invertible.

Solution.

a) False. $(AB)^{-1} = B^{-1}A^{-1}$.

b) True. The first part says the transformation $T(x) = Ax$ is onto. Since A is $n \times n$, then it has n pivots. This is the same as saying A is invertible, and there is no free variable. Therefore, the equation $Ax = b$ has exactly one solution for each b in $\mathbf{R}^n$.

c) True. The first part says the transformation $T(x) = Ax$ is one-to-one (namely not multiple-to-one). Since A is $n \times n$, then it has n pivots. Then there is no free variable. Therefore, the equation $Ax = b$ has exactly one solution for each b in $\mathbf{R}^n$.

d) False. $A + B$ might not be invertible in the first place. For example, if $A = I_2$ and $B = -I_2$ then $A + B = 0$ which is not invertible. Even in the case when $A + B$ is invertible, it still might not be true that $(A + B)^{-1} = A^{-1} + B^{-1}$. For example, $(I_2 + I_2)^{-1} = (2I_2)^{-1} = \frac{1}{2}I_2$, whereas $(I_2)^{-1} + (I_2)^{-1} = I_2 + I_2 = 2I_2$.

e) True. According to the Invertible Matrix Theorem, the product AB is invertible. This implies A is invertible, with inverse $B(AB)^{-1}$:

$$A \cdot B(AB)^{-1} = (AB)(AB)^{-1} = I_n.$$

f) False. In order for Bx to make sense, x must be in $\mathbf{R}^2$, and so Bx is in $\mathbf{R}^4$ and $A(Bx)$ is in $\mathbf{R}^3$. Therefore, the domain of Z is $\mathbf{R}^2$ and the codomain of Z is $\mathbf{R}^3$.

g) True. If the columns of A span $\mathbf{R}^n$, then A is invertible by the Invertible Matrix Theorem. We can also see this directly without quoting the IMT:

If the columns of A span $\mathbf{R}^n$, then A has n pivots, so A has a pivot in each row and column, hence its matrix transformation $T(x) = Ax$ is one-to-one and onto and thus invertible. Therefore, A is invertible.

2. **a)** Given A is a 3×3 invertible matrix, describe how to find A^{-1} using row reduction.

b) Given A, B are both 3×3 matrix, not necessarily invertible, Describe how to find all possible 3×3 matrix X that satisfies $AX = B$.

c) What is the relation between the previous two parts of the question.

Solution.

a) Since A is invertible, we can find inverse by row reduction $\left(A \mid I\right) \to \left(I \mid A^{-1}\right)$.

b) Let's write down notation for columns of B and X explicitly, $B := [b_1, b_2, b_3]$ and $X := [X_1, X_2, X_3]$. Since $AX = [AX_1, AX_2, AX_3]$, then solving $AX = B$ is the same as solving three linear systems

$$AX_i = b_i, \quad i = 1, 2, 3$$

we can do row reduction simultaneously by

$$\left(A \mid B\right) \to \text{RREF: } \left(\widehat{A} \mid \widehat{B}\right)$$

After we obtain RREF, we can write down parametric form for X_i by looking at $\left(\widehat{A} \mid \widehat{B}_i\right)$ where $\widehat{B}_i$ is i-th column of $\widehat{B}$.

c) Part b) is a more general question of part a). Imagine in part b), we change B into 3×3 identity matrix I, then the question become find all X, that satisfies $AX = I$. Since in part a) finding A^{-1} solves $AX = I$ with the condition that A is invertible, which is a more restrictive subproblem of part b).

3. Suppose A is an invertible 3×3 matrix with the following equations hold. Find A.

$$A^{-1}e_1 = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \quad A^{-1}e_2 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \quad A^{-1}e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Solution.

The columns of A^{-1} are

$$(A^{-1}e_1 \ \ A^{-1}e_2 \ \ A^{-1}e_3), \qquad \text{so} \qquad A^{-1} = \begin{pmatrix} 4 & 3 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

To get A, we just find $(A^{-1})^{-1}$. Row-reducing $[A^{-1} \mid I]$ eventually gives us

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{5} & -\frac{3}{5} & 0 \\ 0 & 1 & 0 & -\frac{1}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right), \quad \text{so} \quad A = \begin{pmatrix} \dfrac{2}{5} & -\dfrac{3}{5} & 0 \\ -\dfrac{1}{5} & \dfrac{4}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be rotation *clockwise* by $60°$. Let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation satisfying $U(1,0) = (-2,1)$ and $U(0,1) = (1,0)$.

a) Find the standard matrix for the T and U, and compute the determinant of each matrix.

b) Find the standard matrix for the composition $U \circ T$ using matrix multiplication. Compute the determinant.

c) Find the standard matrix for the composition $T \circ U$ using matrix multiplication. Compute the determinant.

d) Is rotating clockwise by $60°$ and then performing U, the same as first performing U and then rotating clockwise by $60°$?

e) What is the relation between the determinants of these matrices?

Solution.

To reduce confusion on notation, we are going to use T, U to denote standard matrices for linear transformation T, U.

a) The matrix for T is $\begin{pmatrix} \cos(-60°) & -\sin(-60°) \\ \sin(-60°) & \cos(-60°) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$. Its determinant is $\frac{1}{2} * \frac{1}{2} - \frac{\sqrt{3}}{2} * (-\frac{\sqrt{3}}{2}) = \frac{1}{4} + \frac{3}{4} = 1$. (Alternatively we could use the fact that the determinant for a rotation matrix is always 1.)

The matrix for U is $\begin{pmatrix} U(e_1) & U(e_2) \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$. Its determinant is $-2 * 0 - 1 * 1 = -1$.

b) The matrix for $U \circ T$ is

$$\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -1 - \frac{\sqrt{3}}{2} & \frac{1}{2} - \sqrt{3} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

Its determinant is -1, as $\det(UT) = \det(U)\det(T)$

c) The matrix for $T \circ U$ is

$$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 + \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} + \sqrt{3} & -\frac{\sqrt{3}}{2} \end{pmatrix}.$$

Its determinant is -1 also, as $\det(TU) = \det(T)\det(U)$

d) No. In (a) and (b), we found that the standard matrices for $U \circ T$ and $T \circ U$ are different, so the transformations are different.

e) $\det(UT)$ and $\det(TU)$ are the same, since the determinant of the product of two matrices *is* commutative, unlike the product itself. Specifically, $\det(UT) = \det(TU) = \det(T) \times \det(U)$.