Math 1553 Worksheet §3.3, 3.4, and intro to 3.5

Solutions

1. If A is a 3×5 matrix and B is a 3×2 matrix, which of the following are defined?

a) $A - B$

b) AB

c) $A^T B$

d) $B^T A$

e) A^2

Solution.

Only (c) and (d).

a) $A - B$ is nonsense. In order for $A - B$ to be defined, A and B need to have the same number or rows and same number of columns.

b) AB is undefined since the number of columns of A does not equal the number of rows of B.

c) A^T is 5×3 and B is 3×2, so $A^T B$ is a 5×2 matrix.

d) B^T is 2×3 and A is 3×5, so $B^T A$ is a 2×5 matrix.

e) A^2 is nonsense (can't multiply 3×5 with another 3×5).

2. A is $m \times n$ matrix, B is $n \times m$ matrix. Select proper answers from the box. Multiple answers are possible

a) Take any vector x in $\mathbf{R}^m$, then ABx *must be* in:
$\boxed{\mathrm{Col}(A), \quad \mathrm{Nul}(A), \quad \mathrm{Col}(B), \quad \mathrm{Nul}(B)}$

b) Take any vector x in $\mathbf{R}^n$, then BAx *must be* in:
$\boxed{\mathrm{Col}(A), \quad \mathrm{Nul}(A), \quad \mathrm{Col}(B), \quad \mathrm{Nul}(B)}$

c) If $m > n$, then columns of AB could be linearly $\boxed{independent, \quad dependent}$

d) If $m > n$, then columns of BA could be linearly $\boxed{independent, \quad dependent}$

e) If $m > n$ and $Ax = 0$ has nontrivial solutions, then columns of BA could be linearly $\boxed{independent, \quad dependent}$

Solution.

Recall, AB can be computed as A multiplying every column of B. That is $AB = \begin{pmatrix} Ab_1 & Ab_2 & \cdots Ab_m \end{pmatrix}$ where $B = \begin{pmatrix} b_1 & b_2 & \cdots b_m \end{pmatrix}$.

a) $\boxed{\text{Col}(A)}$. Denote $w := Bx$, which is a vector in $\mathbf{R}^n$. $ABx = A(Bx)$ is multiplying A with w which will end up with "linear combination of columns of A". So ABx is in Col(A).

b) $\boxed{\text{Col}(B)}$. Similarly, $BAx = B(Ax)$ is multiplying B with Ax, a vector in R^m, which will end up with "linear combination of columns of B". So BAx is in Col(B).

c) $\boxed{dependent}$. Since $m > n$ means A matrix can have at most n pivots. So $dim(\text{Col}(A)) \leq n$. Notice from first question we know $\text{Col}(AB) \subset \text{Col}(A)$ which has dimension at most n. That means AB can have at most n pivots. But AB is $m \times m$ matrix, then columns of AB must be dependent.

d) $\boxed{independent, dependent}$. Both are possible. Since $m > n$ means B matrix can have at most n pivots. then $\text{Col}(BA) \subset \text{Col}(B)$ means BA can have at most n pivots. Since BA is $n \times n$ matrix, then the columns of BA will be linearly independent when there are n pivots or linearly dependent when there are less than n pivots. Here are two examples.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \text{then } BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \text{then } BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

e) $\boxed{dependent}$. From the second example above, BA has dependent columns, we know "dependent" is one possible answer. Now to see if "independent" is also possible, we need to find out if BA could have n pivots.

Since $Ax = 0$ has nontrivial solution say x^*, then x^* is also a nontrivial solution of $BAx = 0$. That means BA has free variables, and it can not have n pivots. So columns of BA must be linearly dependent.

To summarize what we are actually study here, there are several relations between these subspaces.

$$\text{Col}(AB) \subset \text{Col}(A);$$
$$\text{Col}(BA) \subset \text{Col}(B);$$
$$\text{Nul}(A) \subset \text{Nul}(BA);$$
$$\text{Nul}(B) \subset \text{Nul}(AB);$$

3. Consider the following linear transformations:

$$T : \mathbf{R}^3 \longrightarrow \mathbf{R}^2 \quad T \text{ projects onto the } xy\text{-plane, forgetting the } z\text{-coordinate}$$
$$U : \mathbf{R}^2 \longrightarrow \mathbf{R}^2 \quad U \text{ rotates clockwise by } 90°$$
$$V : \mathbf{R}^2 \longrightarrow \mathbf{R}^2 \quad V \text{ scales the } x\text{-direction by a factor of 2.}$$

Let A, B, C be the matrices for T, U, V, respectively.

a) Compute A, B, and C.

b) Compute the matrix for $V \circ U \circ T$.

c) Compute the matrix for $U \circ V \circ T$.

d) Describe U^{-1} and V^{-1}, and compute their matrices.
If you have not yet seen inverse matrices in lecture, describe geometrically the transformation U^{-1} that would "undo" U in the sense that $(U^{-1} \circ U)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$. Now, do the same for V.

Solution.

a) We plug in the unit coordinate vectors:

$$T(e_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad T(e_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad T(e_3) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
$$U(e_1) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad U(e_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
$$V(e_1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad V(e_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \Longrightarrow \quad C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

b) $CBA = \begin{pmatrix} 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}$.

c) $BCA = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix}$.

d) Intuitively, if we wish to "undo" U, we can imagine that we have rotated a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ by 90° clockwise and we want to return the vector back to its original position of $\begin{pmatrix} x \\ y \end{pmatrix}$. To do this, we need to rotate it 90° *counterclockwise*. Therefore, U^{-1} is counterclockwise rotation by 90°.

Similarly, to undo the transformation V that scales the x-direction by 2, we need to scale the x-direction by $1/2$, so V^{-1} scales the x-direction by a factor of $1/2$.

Their matrices are, respectively,

$$B^{-1} = \frac{1}{0 \cdot 0 - (-1) \cdot 1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$C^{-1} = \frac{1}{2 \cdot 1 - 0 \cdot 0} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}.$$

4. On your computer, go to the Interactive Transformation Challenge! Complete the Zoom, Reflect, and Scale challenges. If you complete a challenge in the optimal number of steps, the interactive demo will congratulate you. See if you can complete each of these challenges in the optimal number of steps.