Math 1553 Worksheet §6.1 - §6.5
Solutions

1. True/False
   (1) If \( u \) is in subspace \( W \), and \( u \) is also in \( W^\perp \), then \( u = 0 \).
   (2) If \( y \) is in a subspace \( W \), the orthogonal projection of \( y \) onto \( W^\perp \) is 0.
   (3) If \( x \) is orthogonal to \( v \) and \( w \), then \( x \) is also orthogonal to \( v - w \).

Solution.
   (1) TRUE: Such a vector \( u \) would be orthogonal to itself, so \( u \cdot u = ||u||^2 = 0 \).

   Therefore, \( u \) has length 0, so \( u = 0 \).
   (2) TRUE: \( y \) is in \( W \), so \( y \perp W^\perp \). Its orthogonal projection onto \( W \) is \( y \) and orthogonal projection onto \( W^\perp \) is 0. In fact \( y \) has orthogonal decomposition \( y = y + 0 \), where \( y \) is in \( W \) and 0 is in \( W^\perp \).
   (3) TRUE: By properties of the dot product, if \( x \) is orthogonal to \( v \) and \( w \) then \( x \) is orthogonal to everything in \( \text{Span}\{v, w\} \) (which includes \( v - w \)).

2. a) Find the standard matrix \( B \) for \( \text{proj}_L \), where \( L = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\} \).

b) What are the eigenvalues of \( B \)? Is \( B \) is diagonalizable? If so, find an invertible \( C \) and diagonal \( D \) so that \( B = CDC^{-1} \)?

c) Describe the column space and null space of the matrix \( B \) in terms of \( L \).

Solution.
   a) We use the formula \( B = \frac{1}{u \cdot u} uu^T \) where \( u = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \) (this is the formula \( B = A(A^TA)^{-1}A^T \) when “A” is just the single vector \( u \)).

\[
B = \frac{1}{1(1) + 1(1) + (-1)(-1)} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}
\]

\[\implies B = \frac{1}{3} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \]

b) \( Bx = x \) for every \( x \) in \( L \), and \( Bx = 0 \) for every \( x \) in \( L^\perp \), so \( B \) has two eigenvalues: \( \lambda_1 = 1 \) with algebraic (and geometric) multiplicity 1, \( \lambda_2 = 0 \) with algebraic (and geometric) multiplicity 2. Here \( v_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \) is an eigenvector for \( \lambda_1 = 1 \), whereas \( v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \) and \( v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \) are eigenvectors for \( \lambda_2 = 0 \).
Therefore

\[
B = \begin{pmatrix}
1 & 1 & 1 \\
1 & -1 & 0 \\
-1 & 0 & 1 \\
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
1 & -1 & 0 \\
\end{pmatrix}^{-1}
\]

c) \( \text{Col}(B) = L \) and \( \text{Nul}(B) = L^\perp \).

3. \( y = \begin{pmatrix}
0 \\
2 \\
4 \\
\end{pmatrix} \), \( u_1 = \begin{pmatrix}
1 \\
1 \\
0 \\
\end{pmatrix} \), \( u_2 = \begin{pmatrix}
-1 \\
1 \\
0 \\
\end{pmatrix} \)

1) Determine whether \( u_1 \) and \( u_2 \)
   (a) are linearly independent
   (b) are orthogonal
   (c) span \( \mathbb{R}^3 \)

2) Is \( y \) in \( W = \text{Span}\{u_1, u_2\} \)?

3) Compute the vector \( w \) that most closely approximates \( y \) within \( W \).

4) Construct a vector, \( z \), that is in \( W^\perp \).

5) Make a rough sketch of \( W, y, w, \) and \( z \).

Solution.

1) A quick check shows that the vectors \( u_1 \) and \( u_2 \) are orthogonal and linearly independent, so \( \text{Span}\{u_1, u_2\} \) is a plane in \( \mathbb{R}^3 \), but is not all of \( \mathbb{R}^3 \).

2) By inspection, \( y \) is not in the span because it has a non-zero \( x_3 \) component.

3) The vector \( w \) is \( \text{proj}_W y \). The orthogonal projection of \( y \) onto \( W \) is calculated in the usual way.

\[
A^T Av = A^T b
\]

\[
A^T = \begin{pmatrix}
2 & 0 \\
0 & 2 \\
\end{pmatrix}, \quad A^T b = \begin{pmatrix}
2 \\
2 \\
\end{pmatrix}, \quad \text{so} \quad \begin{pmatrix}
2 & 0 \\
0 & 2 \\
\end{pmatrix} v = \begin{pmatrix}
2 \\
2 \\
\end{pmatrix}, \quad v = \begin{pmatrix}
1 \\
1 \\
\end{pmatrix}
\]

\[
w = Av = \begin{pmatrix}
1 & -1 \\
1 & 1 \\
0 & 0 \\
\end{pmatrix} \begin{pmatrix}
1 \\
1 \\
\end{pmatrix} = \begin{pmatrix}
0 \\
2 \\
0 \\
\end{pmatrix}.
\]

Another quick way to do this problem is note that \( W \) is the \( xy \)-plane of \( \mathbb{R}^3 \), so the projection of \( \begin{pmatrix}
0 \\
2 \\
4 \\
\end{pmatrix} \) onto \( W \) is just \( \begin{pmatrix}
0 \\
2 \\
0 \\
\end{pmatrix} \).

4) One vector in \( W^\perp \) is \( z = y - \text{proj}_W y = \begin{pmatrix}
0 \\
2 \\
4 \\
\end{pmatrix} - \begin{pmatrix}
0 \\
2 \\
4 \\
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
\end{pmatrix} \).

5) Here [https://www.geogebra.org/calculator/](https://www.geogebra.org/calculator/) is a picture you can play with. The vector \( w \) is labeled “\( \text{\textcopyright} \) in the drawing.
4.  a) Find the least squares solution $\hat{x}$ to $Ax = e_1$, where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}$.

b) Find the best fit line $y = Ax + B$ through the points $(0, 0)$, $(1, 8)$, $(3, 8)$, and $(4, 20)$.

c) Set up an equation to find the best fit parabola $y = Ax^2 + Bx + C$ through the points $(0, 0)$, $(1, 8)$, $(3, 8)$, and $(4, 20)$.

**Solution.**

a) We need to solve the equation $A^T A \hat{x} = A^T e_1$. We compute:

$$A^T A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix},$$

$$A^T e_1 = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Now we form the augmented matrix:

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/3 \end{pmatrix} \Rightarrow \hat{x} = \begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix}.$$

b) We want to find a least squares solution to the system of linear equations

$$
\begin{align*}
0 &= A(0) + B \\
8 &= A(1) + B \\
8 &= A(3) + B \\
20 &= A(4) + B
\end{align*}
\iff
\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 8 \end{pmatrix}.$$
We compute
\[
\begin{pmatrix} 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 26 & 8 \\ 8 & 4 \end{pmatrix}
\]
\[
\begin{pmatrix} 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} = \begin{pmatrix} 112 \\ 36 \end{pmatrix}
\]
\[
\begin{pmatrix} 26 & 8 & 112 \\ 8 & 4 & 36 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \end{pmatrix}.
\]
Hence the least squares solution is \(A = 4\) and \(B = 1\), so the best fit line is \(y = 4x + 1\).

c) We want to find a least squares solution to the system of linear equations
\[
\begin{align*}
0 &= A(0^2) + B(0) + C \\
8 &= A(1^2) + B(1) + C \\
8 &= A(3^2) + B(3) + C \\
20 &= A(4^2) + B(4) + C
\end{align*}
\]
\[
\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}.
\]
We compute
\[
\begin{pmatrix} 0 & 1 & 9 & 16 \\ 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 338 & 92 & 26 \\ 92 & 26 & 8 \\ 26 & 8 & 4 \end{pmatrix}
\]
\[
\begin{pmatrix} 0 & 1 & 9 & 16 \\ 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} = \begin{pmatrix} 400 \\ 112 \\ 36 \end{pmatrix}
\]
\[
\begin{pmatrix} 338 & 92 & 26 & 400 \\ 92 & 26 & 8 & 112 \\ 26 & 8 & 4 & 36 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 4/3 \\ 0 & 0 & 1 & 2 \end{pmatrix}.
\]
Hence the least squares solution is \(A = 2/3\), \(B = 4/3\), and \(C = 2\), so the best fit quadratic is \(y = \frac{2}{3}x^2 + \frac{4}{3}x + 2\).

There is a picture on the next page. The “best fit cubic" would be the cubic \(y = \frac{5}{3}x^3 - \frac{28}{3}x^2 + \frac{47}{3}x\), which actually passes through all four points. One can fit the points with even higher order polynomials.
\begin{align*}
y &= 4x + 1 \\
y &= \frac{2}{3}x^2 + \frac{4}{3}x + 2 \\
y &= \frac{5}{3}x^3 - \frac{28}{3}x^2 + \frac{47}{3}x
\end{align*}