Math 1553 Worksheet: Fundamentals and §1.1

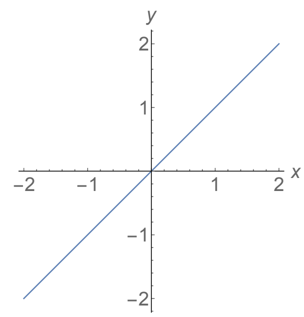
Solutions

1. **a)** (Warm-up) Draw the set of all points in $\mathbf{R}^2$ that satisfy the equation $x-y=0$, where we use (x,y) to denote points in $\mathbf{R}^2$.

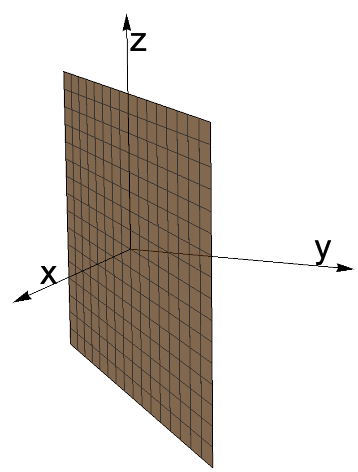
b) Draw the set of all points in $\mathbf{R}^3$ that satisfy the equation $x-y=0$, where we use (x,y,z) to denote points in $\mathbf{R}^3$. Geometrically, does this set form a line, a plane, or something else?

Solution.

a) This is the line $y=x$.



b) Adding y to both sides gives $y=x$ like in part (a). However, points in $\mathbf{R}^3$ have three coordinates (x,y,z) rather than just two coordinates, so we will get a *plane* rather than a line: $y=x$ but z can be anything we want. In other words, the points in $\mathbf{R}^3$ satisfying $x-y=0$ are the points (x,x,z) where x and z are any real numbers.



2. Richard Straker has eight light switches in order along a wall. He records which lights are on and which lights are off. To save time, he uses 0 to represent "off" and using 1 to represent "on" for each light.

a) Write an element of $\mathbf{R}^n$ (for some n) that represents the situation when all the lights are on. What is n?

b) Repeat part (a) when all lights are off.

Solution.

a) If all eight lights are on, then each gets a 1, so we can represent this by $(1,1,1,1,1,1,1,1)$, which is in $\mathbf{R}^8$ because it has eight coordinates (one for each light).

b) If all eight lights are off, then each gets a 0, so we can represent this by $(0,0,0,0,0,0,0,0)$, which is in $\mathbf{R}^8$.

3. **a)** (Warm-up) In how many ways can two lines in the xy-plane intersect? Draw a quick picture for each case.

b) Is it possible for two planes in $\mathbf{R}^3$ to intersect in a line? If so, draw an example. Can you write a system of two equations that represents this?

c) Is it possible for the intersection of two planes in $\mathbf{R}^3$ to consist of exactly one point? If so, draw an example. Can you write a system of two equations that represents this?

d) Is it possible for the intersection of three planes in $\mathbf{R}^3$ to be exactly one point? If so, draw an example. Can you write a system of three equations that represents this?

Solution.

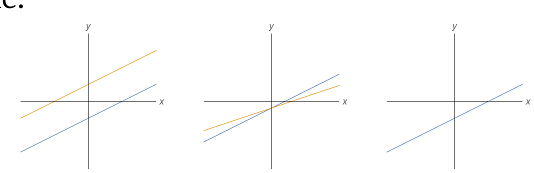
a) There are three possibilities.

(1) The lines do not intersect at all. This occurs when they are different parallel lines. For example, the lines $y = x + 1$ and $y = x - 2$ never intersect. If they intersected, then there would be a solution to the system of equations below which is clearly inconsistent (it gives $-1 = 2$).

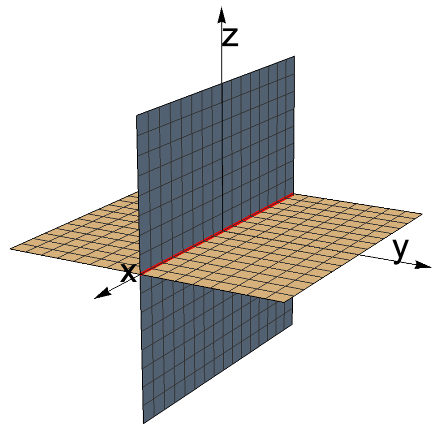
$$x - y = -1$$
$$x - y = 2.$$

(2) The lines intersect at exactly one point.

(3) The lines intersect at infinitely many points. This occurs when they are the same line.



b) Yes. For example, consider the xz-plane ($y = 0$) and the xy-plane ($z = 0$). They intersect along the x-axis, as illustrated below.



We could also see that their intersection is the x-axis without drawing a picture: the requirement for a point to be in both of these planes is for $y = 0$ and $z = 0$, so x can be anything we want, i.e. the point is $(x, 0, 0)$. The system of equations is just

$$y = 0$$
$$z = 0,$$

or less succinctly:

$$0x + y + 0z = 0$$
$$0x + 0y + z = 0.$$

c) No. To see this intuitively, imagine what happens if the planes intersected at just one point. In order to only touch at that point, the planes would need to somehow curve away from each other near that point, which is impossible since planes are completely flat. Later in chapter 1, we will use row reduction and the concept of “free variables” to see algebraically why this is the case.

d) Yes. To keep it as simple as possible, consider the coordinate planes: the yz-plane ($x = 0$), the xz-plane ($y = 0$), and the xy-plane ($z = 0$). For a point to be in all three of these planes, we need $x = 0$ and $y = 0$ and $z = 0$, which gives us only the point $(0, 0, 0)$. This is actually very difficult to picture because drawing multiple planes gets messy. The system of equations is:

$$x = 0$$
$$y = 0$$
$$z = 0,$$

or in longhand notation:

$$x + 0y + 0z = 0$$
$$0x + y + 0z = 0$$
$$0x + 0y + z = 0.$$

4. For each equation, determine whether the equation is linear or non-linear. Circle your answer. If the equation is non-linear, briefly justify why it is non-linear.

a) $3x_1 + \sqrt{x_2} = 4$ Linear Not linear

b) $x^2 + y = z$ Linear Not linear

c) $e^\pi x + \ln(13)y = \sqrt{2} - z$ Linear Not linear

Solution.

a) Not linear. The $\sqrt{x_2}$ term makes it non-linear.

b) Not linear. It has a quadratic term x^2.

c) Linear. Don't be fooled: e^π and $\ln(13)$ are just the coefficients for x and y, respectively, and $\sqrt{2}$ is a constant term.

If, for example, the second term had been $\ln(13y)$ instead of $\ln(13)y$, then y would have been inside the logarithm and the equation would have been non-linear.