Math 1553 midterm exam 1

Solutions

1. Solution.

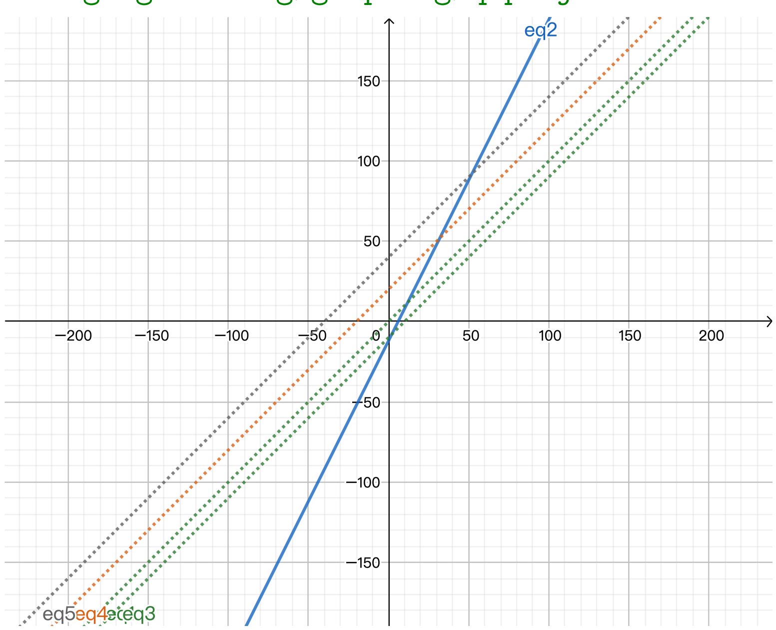
a) $\left(\begin{array}{ccc|c} 1 & 0 & \color{blue}{2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ is not in RREF, since $\color{blue}{2}$ is above the pivot which is not zero.

b) $\left(\begin{array}{cccc|c} 1 & 0 & 0 & 5 & -7 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$. There are two pivots, and there are two free variables x_3, x_4.

2. Solution.

a) If an augmented matrix has a pivot in every column, then the pivot in the last column (on the right of the vertical bar) has a pivot. That implies the linear system is inconsistent.

b) The two lines in $\mathbf{R}^2$: $x - y = c$ and $2x - y = 12$ have different slopes. So they will intersect no matter what the intercept c is.
From linear system point of view, we need to have a consistent linear system to have the two lines intersect. One can write down augmented matrix, $\left(\begin{array}{cc|c} 1 & -1 & c \\ 2 & -1 & 12 \end{array}\right)$, and do row reduction find REF to be $\left(\begin{array}{cc|c} 1 & -1 & c \\ 0 & 1 & 12-2c \end{array}\right)$ which is consistent for any c. Here is an online graph you can play with https://www.geogebra.org/graphing/qapz8jts



3. Solution.

a) The last row of $\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right)$ is $\left(\begin{array}{ccc|c} 0 & 0 & 0 & 2 \end{array}\right)$, which yields a contradiction.

b) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$ is consistent and have no free variables. So the linear system has a unique solution $x_1 = 0, x_2 = 0, x_3 = -2$.

4. Solution.

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \quad b = \begin{pmatrix} 8 \\ 5 \\ -8 \end{pmatrix}$$

We want to find x_1, x_2 such that $x_1 v_1 + x_2 v_2 = b$. This is a linear system, we can solve it using row reduction.

$$\left(\begin{array}{cc|c} 1 & 2 & 8 \\ 1 & 1 & 5 \\ -1 & -2 & -8 \end{array}\right) \xrightarrow[R_3=R_3+R_1]{R_2=R_2-R_1} \left(\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow[R_2=-R_2]{R_1=R_1+2R_2} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right)$$

This gives us $x_1 = 2, x_2 = 3$.

5. Solution.

a) $u = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \quad v = \begin{pmatrix} 3 \\ 3/2 \\ -3/2 \end{pmatrix}$. We find $v = \frac{3}{2}u$, which means u, v are on the same line. So the Span$\{u, v\}$ must be a line.

b) It is nonlinear since there is a cross product term $x_1 x_2$.

6. Solution.

Most of these questions can be converted to counting the number of pivots, and free variables.

a) A is a 3×3 matrix, and if $\left(A \mid b_1\right)$ has a unique solution, that means there is no free variable, and there are 3 pivots in the coefficient matrix A. So Every row and every column of A has exactly one pivot. Then for any b_2, the $\left(A \mid b_2\right)$ will exactly have only one solution.

b) Same as last part, A is 3×3, and if $\left(A \mid b_1\right)$ has a unique solution, that means there are 3 pivots in the coefficient matrix A. So every row and every column of A has exactly one pivot. for any b_2, the $\left(A \mid b_2\right)$ will exactly have only one solution.

7. Solution.

a) To be inconsistent, we need the two lines to be parallel. So we need the line $x - hy = 10$ to have the same slope as the line $3x + 6y = k$, which means we want $h = -2$. Then we need to make sure that the two lines are not the *same* line, so we want our constant term to be different. As long as $k \neq 30$, we have that $x + 2y = 10$ and $3x + 6y = k$ are two different parallel lines, and which makes this system inconsistent.

b) To have exactly one solution, all that matters is that the two lines are *not* parallel, so so long as $h \neq -2$, the two lines have different slopes and so must intersect at exactly one point.

8. Solution.

To check whether $\text{Span}\{u, v, w\} = \mathbb{R}^3$, we could build a matrix with u, v, and w as columns and row reduce it to check whether it has three pivots. We would notice that it has only two pivots, and so the span is a plane and not all of $\mathbb{R}^3$. Alternatively, we could notice that $u + v = -w$, meaning that the set $\{u, v, w\}$ is linearly dependent, and the matrix will have a free variable. Since it would be a 3x3 matrix, that means its column span is at most 2-dimensional and not all of $\mathbb{R}^3$.

9. Solution.

a) The solution set of a homogeneous equation is always a span, which this isn't. To pick a concrete example, the vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is always a solution to $Ax = 0$, when A is a $m \times 3$ matrix, but that vector is not in the proposed solution set.

b) The solution set of $Ax = b$, when A is a $m \times 3$ matrix, must be a set of vectors in $\mathbf{R}^3$. Specifically, notice that Ax, when A is a 4×3 matrix and x is 4×1 vector is not a legal multiplication.

10. Solution.

a) Let $x_1 = c_1$ and $x_2 = c_2$ be the unique solution we are promised to the equation $x_1 v_1 + x_2 v_2 = w$. If $\text{Span}\{v_1, v_2\}$ isn't a plane, then the set $\{v_1, v_2\}$ is dependent, and there is a nontrivial solution to $x_1 v_1 + x_2 v_2 = 0$, call that $x_1 = a_1$ and $x_2 = a_2$. Then we can simply add the two solutions together to get

$$(c_1 + a_1)v_1 + (c_2 + a_2)v_2 = w,$$

a different solution to $x_1 v_1 + x_2 v_2 = w$. Since this is impossible, v_1 and v_2 must be independent, and so $\text{Span}\{v_1, v_2\}$ must be a plane.

b) If $x_1 v_1 + x_2 v_2 = w$ has unique solution $x_1 = c_1$ and $x_2 = c_2$, then $c_1 v_1 + c_2 v_2 + (-1)w = 0$ is a nontrivial solution to $x_1 v_1 + x_2 v_2 + x_3 w = 0$, and so the set $\{v_1, v_2, w\}$ must be dependent.

11. Solution.

First notice that $\{u, 10v, 3u - 4v\}$ is linearly dependent, since $3u - 4v = 3u + -\frac{4}{10}(10v)$, and so we only need to consider if $\text{Span}\{u, v\} = \text{Span}\{u, 10v\}$. Then it is clear that any vector $c_1u + c_2v$ in $\text{Span}\{u, v\}$ can be simply rewritten $c_1u + c_2v = c_1u + \frac{c_2}{10}(10v)$, and so is in $\text{Span}\{u, 10v\}$, and likewise any vector $d_1u + d_2(10v)$ in $\text{Span}\{u, 10v\}$ can be rewritten $d_1u + d_2(10v) = d_1u + (10d_2)v$, and so is in $\text{Span}\{u, v\}$.

12. Solution.

If b is in the column span of a matrix A, then by definition $Ax = b$ is consistent. However, the difference between $Ax = b$ having one solution and infinitely many depends on the number of free variables that A has. A matrix is only guaranteed free variables once we have more columns than rows, so we cannot say either way here.

13. **a)** A is a 4×5 matrix, so the solution set to $Ax = 0$ lives in $\mathbf{R}^5$. We are told A has 5 columns but only 4 pivots, so the augmented matrix $\left(A \mid 0\right)$ will have one column left of the augment bar without a pivot (i.e. one free variable). Therefore, the solution set is a line in $\mathbf{R}^5$.

b) A is a 4×5 matrix, so its column span lives in $\mathbf{R}^4$. Since we are told that A has 4 pivots, this means that A has a pivot in every row, so for every b in $\mathbf{R}^4$, the equation $Ax = b$ is consistent. Since A has 5 columns but only 4 pivots, the (consistent) system $Ax = b$ will have a free variable and therefore infinitely many solutions.

14. The blue line given to us is $\text{Span}\left\{\begin{pmatrix}1\\-2\end{pmatrix}\right\}$, and the matrices are

$$A = \begin{pmatrix}-2 & 0\\4 & 0\end{pmatrix}, \quad B = \begin{pmatrix}1 & -2\\1 & -2\end{pmatrix}, \quad C = \begin{pmatrix}1 & 2\\-2 & -4\end{pmatrix}.$$

We see A and C each have $\text{Span}\left\{\begin{pmatrix}1\\-2\end{pmatrix}\right\}$ as their column span (this is the line $x_2 = -2x_1$ in $\mathbf{R}^2$), but B does not.

15. **a)** This is true. This T/F problem is just a rephrasing of the fact that if A is an $m \times n$ matrix with more columns than rows (i.e. a "wide" matrix), then the columns of A cannot be linearly independent.

If $\{v_1, v_2, v_3, v_4, v_5\}$ is a linearly independent set of vectors in $\mathbf{R}^n$, then it must be true that $n \geq 5$. For example, if $n = 4$ then this is a set of 5 vectors in $\mathbf{R}^4$, so it cannot be linearly independent.

b) This is false. The vector equation

$$x_1w_1 + \cdots + x_pw_p = 0$$

ALWAYS has the trivial solution $x_1 = \cdots = x_p = 0$. The vectors $\{w_1, \dots, w_p\}$ are linearly independent when the trivial solution is the ONLY solution.

16. **a)** This is false. If the columns of an $m \times n$ matrix A are linearly dependent, then the matrix equation $Ax = b$ might still be consistent for every b in $\mathbf{R}^m$. For example, take the 2×3 matrix from Quiz 3:

$$A = \begin{pmatrix} 2 & -3 & -1 \\ 0 & 0 & 5 \end{pmatrix}.$$

The columns of A are linearly dependent, but A has a pivot in every row, so $Ax = b$ is consistent for each b in $\mathbf{R}^2$.

b) This is true. If $\{u_1, u_2, u_3, u_4\}$ are linearly independent vectors in $\mathbf{R}^4$, then the 4×4 matrix A whose columns are u_1 through u_4 will have 4 pivots and therefore a pivot in every row, so u_1 through u_4 must span all of $\mathbf{R}^4$.

17. **a)** $Au = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $Av = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so

$$A(3u - 2v) = 3Au - 2Av = \begin{pmatrix} 12 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \end{pmatrix}.$$

b) Yes. Here $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, so for any b in $\mathbf{R}^2$, the vector equation $x_1 u + x_2 v = b$ corresponds to the augmented matrix $\left(\begin{array}{cc|c} 1 & -3 & (*)_1 \\ 0 & 1 & (*)_2 \end{array}\right)$. We see from the pivots that this system is consistent and has a unique solution.

18. Suppose A is a 3×4 matrix and b is some vector so that the solution set for the matrix equation $Ax = b$ has parametric vector form

$$\begin{pmatrix} 4 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -4 \\ 0 \\ -2 \\ 1 \end{pmatrix},$$

where x_2 and x_4 can be any real numbers.

This problem uses the fact that the solution set for $Ax = b$ is the translation of the solution set of $Ax = 0$. In the parametric vector form, the term with no free variable is a particular solution p (where $Ap = b$) and the terms with the free variables give the homogeneous solutions ($Ax = 0$).

a) Yes, it is true that $A\begin{pmatrix}4\\0\\1\\0\end{pmatrix} = b$. We know this already from the reasoning above, but we could see it directly by noting that when $x_2 = 0$ and $x_4 = 0$, the solution to $Ax = b$ that we get is just $\begin{pmatrix}4\\0\\1\\0\end{pmatrix}$.

b) Yes, it is true that $A\begin{pmatrix}-4\\0\\-2\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$. From the parametric vector form above, we see that in fact $\begin{pmatrix}-4\\0\\-2\\1\end{pmatrix}$ was given to us as one of the vectors in the spanning set for the homogeneous solutions.

19. We need to write a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

that satisfies both the following properties:

(1) The span of the columns of A is the line $x_1 = 2x_2$.
(2) The set of solutions to the equation $Ax = 0$ is Span$\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\}$

Solution:
(1) The column span consists of all vectors of the form

$$\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}2x_2\\x_2\end{pmatrix} = x_2\begin{pmatrix}2\\1\end{pmatrix}, \text{ namely } \mathrm{Span}\left\{\begin{pmatrix}2\\1\end{pmatrix}\right\}.$$

Therefore, each column of A should be a scalar multiple of $\begin{pmatrix}2\\1\end{pmatrix}$.

(2) The span of $\begin{pmatrix}3\\1\end{pmatrix}$ is all vectors where $x_1 = 3x_2$, which gives us the equation $x_1 - 3x_2 = 0$, whose augmented form is $\left(\begin{array}{cc|c} 1 & -3 & 0 \end{array}\right)$. From this, we see the second column of A should be -3 times the first column.

Putting these together, we could write

$$A = \begin{pmatrix}2 & -6\\1 & -3\end{pmatrix}.$$

For this A, we have
$a = 2$,
$b = -6$,

$c = 1,$
$d = -3.$

There are other possibilities for A, for example

$$A = \begin{pmatrix} 4 & -12 \\ 2 & -6 \end{pmatrix}, \qquad A = \begin{pmatrix} -2 & 6 \\ -1 & 3 \end{pmatrix}, \qquad \text{etc..}$$