Math 1553 midterm exam 3

Solutions

1. Honor code

2. The vector from $(1,0)$ to $(4,5)$ is $(3,5)$ and the vector from $(1,0)$ to $(1,-4)$ is $(0,-4)$. So the area of the triangle is

$$\frac{1}{2}\left|\det\begin{pmatrix}3 & 0\\ 5 & -4\end{pmatrix}\right| = \frac{1}{2}(12) = 6.$$

3. The columns of the matrix are linearly dependent (in fact, the first three columns are identical!), so its determinant is 0.

4. We solve

$$\det\begin{pmatrix}1 & 0 & 4\\ 0 & c & -5\\ 1 & 3 & 7\end{pmatrix} = 3$$

$$7c + 15 - 4c = 3, \qquad 3c = -12, \qquad c = -4.$$

5. Taken from a worksheet. If A is $n \times n$ then $\det(cA) = c^n \det(A)$. Here A is 3×3, so

$$\det(2A) = 2^3 \det(A) = 8 \det(A).$$

6. We are told that A is 5×5 and $\det(A) = 3$.

a) True. The columns of A form a basis for $\mathbf{R}^n$, since A is invertible.

b) True. The columns of A are linearly independent since A is invertible.

c) False. The rank of A is 5 since A is invertible.

d) True. The null space of A is just the zero vector, since $Ax = 0$ has only the trivial solution.

7. Copied from a worksheet.

a) The correct answer is (III). This was copied from one of our chapter 5 worksheets.

b) The correct answer is (III). This was copied from one of our chapter 5 worksheets.

8. **a)** Since A has $\lambda = -1$ as an eigenvalue, the equation $(A + I)x = 0$ has infinitely many solutions since $Ax = -x$ has a non-trivial solution.

b) $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 3$, and to get the matrix below requires a row swap and multiplying a row by -2, so

$$\det\begin{pmatrix} -2c & -2d \\ a & b \end{pmatrix} = 3(-1)(-2) = 6.$$

9. $A = \begin{pmatrix} 7 & 4 & 4 \\ 4 & 7 & 4 \\ 0 & 0 & 4 \end{pmatrix}$ so

$$(A - 3I|0) = \left(\begin{array}{ccc|c} 4 & 4 & 4 & 0 \\ 4 & 4 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This gives $x_1 + x_2 = 0$, x_2 free, and $x_3 = 0$, so a basis for the 3-eigenspace is $\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right\}$.

10.

a) True. The matrix A gives counterclockwise rotation by $23°$, which means that if $v \neq 0$, then v and Av will not be on the same line through the origin. Therefore, A doesn't have any real eigenvalues.

b) True: u and v are eigenvectors for $\lambda = 2$ and $u + v$ is not the zero vector, so $u+v$ is also a 2-eigenvector. You can see this by recalling that the 2-eigenspace is a subspace (thus closed under addition), or note

$$A(u + v) = Au + Av = 2u + 2v = 2(u + v).$$

11. Taken from the Webwork and a quiz. $A = \begin{pmatrix} 1 & k \\ 1 & 3 \end{pmatrix}$, so its char. polynomial is

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - 4\lambda + 3 - k.$$

This has one real eigenvalue of algebraic multiplicity 2 precisely when the polyomial is a square, so it equals

$$(\lambda - 2)^2 = \lambda^2 - 4\lambda + 4,$$

thus $3 - k = 4$ so $k = -1$.

12. We expand the characteristic polynomial along the third row: $A = \begin{pmatrix} 1 & 4 & -1 \\ 2 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ so

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 4 & -1 \\ 2 & 3-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{pmatrix} = (-1)^6(1-\lambda)\Big[(1-\lambda)(3-\lambda) - 8\Big]$$
$$= (1-\lambda)(\lambda^2 - 4\lambda - 5) = (1-\lambda)(\lambda - 5)(\lambda + 1).$$

The eigenvalues are $\lambda = -1$, $\lambda = 1$, $\lambda = 5$.

13. **a)** True. $\det(A-\lambda I) = -\lambda^3 - 4\lambda^2 = -\lambda^2(\lambda+4)$, so if the 0-eigenspace is a plane then the matrix is automatically diagonalizable because the sum of geometric multiplicities of $\lambda = 0$ and $\lambda = -4$ is then automatically $2+1=3$.

b) Need more information. We know A is 6×6 with exactly 4 real eigenvalues, but we are only told that (at least) one of the eigenvalues has geometric multiplicity 2, so this means the sum of geometric multiplicities is 5 or 6. If another eigenvalue has geo. mult. 2, then A is diagonalizable. However, if the rest each only have geo. mult. 1, then A is not diagonalizable.

14. **a)** True. Copied from a worksheet.

b) True:

$$\det(A) = \det(CDC^{-1}) = \det(C)\det(D)\det(C^{-1}) = \det(C)\cdot\det(D)\cdot\frac{1}{\det(C)} = \det(D).$$

15. **a)** $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is not diagonalizable.

Its only eigenvalue is $\lambda = 1$, but $\mathrm{Nul}(A-I)$ gives only two free variables, so the 1-eigenspace only has dimension 2.

b) Yes, B is a 2×2 matrix with two real eigenvalues $\lambda = 1$ and $\lambda = -1$, so B is diagonalizable.

16. Since $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is in the 1-eigenspace and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is in the 2-eigenspace, we get

$$A\left(\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right) = A\begin{pmatrix} 4 \\ 1 \end{pmatrix} + A\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix}.$$

So $k = 5$.

17. We are told the 2×2 matrix A has eigenvalue $\lambda_1 = -2 + i\sqrt{5}$ and corresponding eigenvector $\begin{pmatrix} 10 \\ -5 - i\sqrt{5} \end{pmatrix}$.

a) Complex eigenvalues come in complex conjugate pairs, so $\lambda_2 = -2 - i\sqrt{5}$ is its other eigenvalue.

b) We get an eigenvector for $\lambda = 2$ by taking the complex conjugate of each entry of the eigenvector for λ_1, which gives us $\begin{pmatrix} 10 \\ -5 + i\sqrt{5} \end{pmatrix}$.

18. The positive 2×2 stochastic matrix A has 1-eigenspace spanned by $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, so its steady-state vector is

$$w = \frac{1}{1+2}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}.$$

Here $v = \begin{pmatrix} 120 \\ 30 \end{pmatrix}$. By the Perron-Frobenius Theorem, we know that as n gets very large, $A^n v$ approaches

$$(120+30)w = 150\begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} = \begin{pmatrix} 50 \\ 100 \end{pmatrix}.$$

19. A is a positive stochastic 3×3 matrix.

a) True, there is exactly one steady-state vector for A by the Perron-Frobenius Theorem.

b) True. Each column sums to 1, and there are three columns, so the sum of all entries in the matrix is 3.

20. **a)** True. If A is 7×7 then it must have at least one real eigenvalue. Since (non-real) complex eigenvalues (and their powers) come in conjugate pairs, only an "even" $\times$ "even" matrix A can have no real eigenvalues.

Alternatively: since $\det(A - \lambda I)$ is a degree 7 polynomial, it has at least one real root just due to a precalculus argument using end-behavior and continuity of polynomial functions.

b) True. If $Av = \lambda v$ then we know

$$A^2 v = A(\lambda v) = \lambda A v = \lambda^2 v$$

This means λ^2 and v is a pair of eigenvalue and eigenvector for A^2.

21. This problem is a simplified version of a problem from the supplemental problems in 5.1-5.2.

$A = \begin{pmatrix} 3 & c \\ 2 & 1 \end{pmatrix}$ and we need $\lambda = 2$ to be an eigenvalue. This is the same as $A - 2I$ is not invertible. We row-reduce

$$(A - 2I \mid 0) = \left(\begin{array}{cc|c} 1 & c & 0 \\ 2 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & c & 0 \\ 0 & -1-2c & 0 \end{array}\right)$$

Since $A - 2I$ is not invertible, we have $-1-2c = 0$, so $c = -1/2$. Alternatively, we could have solved for $\det(A - 2I) = 0$ and found $c = -1/2$.