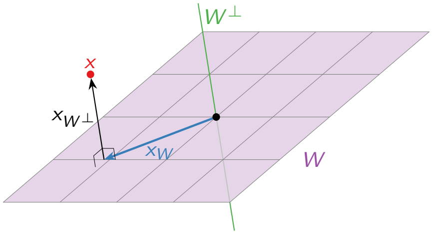
Orthogonal Projections

Review of 6.3 so far

Recall: Let W be a subspace of $\mathbf{R}^n$.

- The **orthogonal complement** $W^\perp$ is the set of vectors orthogonal to everything in W.
- The **orthogonal decomposition** of a vector x with respect to W is the unique way of writing $x = x_W + x_{W^\perp}$ for x_W in W and $x_{W^\perp}$ in $W^\perp$.
- The vector x_W is the **orthogonal projection** of x onto W. It is the closest vector to x in W.
- To compute x_W, write W as $\operatorname{Col} A$ and solve $A^T A v = A^T x$; then $x_W = Av$.



Projection as a Transformation

Change in Perspective: let us consider orthogonal projection as a *transformation*.

Definition

Let W be a subspace of $\mathbf{R}^n$. Define a transformation

$$T\colon \mathbf{R}^n \longrightarrow \mathbf{R}^n \quad \text{by} \quad T(x) = x_W.$$

This transformation is also called **orthogonal projection** with respect to W.

Theorem

Let W be a subspace of $\mathbf{R}^n$ and let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be the orthogonal projection with respect to W. Then:

1. T is a *linear* transformation.
2. For every x in $\mathbf{R}^n$, $T(x)$ is the *closest* vector to x in W.
3. For every x in W, we have $T(x) = x$.
4. For every x in $W^\perp$, we have $T(x) = 0$.
5. $T \circ T = T$.
6. The range of T is W and the null space of T is $W^\perp$.

Projection Matrix

Method 1

Let W be a subspace of $\mathbf{R}^n$ and let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be the orthogonal projection with respect to W.

Since T is a linear transformation, it has a matrix. How do you compute it?

The same as any other linear transformation: compute $T(e_1), T(e_2), \ldots, T(e_n)$.

Problem: Let $L = \mathrm{Span}\left\{\binom{3}{2}\right\}$ and let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be the orthogonal projection onto L. Compute the matrix A for T.

It's easy to compute orthogonal projection onto a line:

$$\left.\begin{aligned} T(e_1) &= (e_1)_L = \frac{u\cdot e_1}{u\cdot u}\,u = \frac{3}{13}\begin{pmatrix}3\\2\end{pmatrix} \\ T(e_2) &= (e_2)_L = \frac{u\cdot e_2}{u\cdot u}\,u = \frac{2}{13}\begin{pmatrix}3\\2\end{pmatrix}\end{aligned}\right\} \quad\Longrightarrow\quad A = \frac{1}{13}\begin{pmatrix}9 & 6\\ 6 & 4\end{pmatrix}.$$

Projection Matrix
Another Example

Problem: Let

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x_1 - x_2 + x_3 = 0 \right\}$$

and let $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ be orthogonal projection onto W. Compute the matrix B for T.

In the slides for the last lecture we computed $W = \operatorname{Col} A$ for

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

To compute $T(e_i)$ we have to solve the matrix equation $A^T A v = A^T e_i$. We have

$$A^T A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad A^T e_i = \text{the } i\text{th column of } A^T = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Projection Matrix

Another Example, Continued

Problem: Let

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x_1 - x_2 + x_3 = 0 \right\}$$

and let $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ be orthogonal projection onto W. Compute the matrix B for T.

$$\left(\begin{array}{cc|c} 2 & -1 & 1 \\ -1 & 2 & -1 \end{array}\right) \overset{\text{RREF}}{\rightsquigarrow} \left(\begin{array}{cc|c} 1 & 0 & 1/3 \\ 0 & 1 & -1/3 \end{array}\right) \implies T(e_1) = \frac{1}{3}A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & -1 & 1 \\ -1 & 2 & 0 \end{array}\right) \overset{\text{RREF}}{\rightsquigarrow} \left(\begin{array}{cc|c} 1 & 0 & 2/3 \\ 0 & 1 & 1/3 \end{array}\right) \implies T(e_2) = \frac{1}{3}A\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & -1 & 0 \\ -1 & 2 & 1 \end{array}\right) \overset{\text{RREF}}{\rightsquigarrow} \left(\begin{array}{cc|c} 1 & 0 & 1/3 \\ 0 & 1 & 2/3 \end{array}\right) \implies T(e_2) = \frac{1}{3}A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$\implies B = \frac{1}{3}\begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}.$$

Theorem

Let $\{v_1, v_2, \ldots, v_m\}$ be a *linearly independent* set in $\mathbf{R}^n$, and let

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{pmatrix}.$$

Then the $m \times m$ matrix $A^T A$ is invertible.

Proof: We'll show $\mathrm{Nul}(A^T A) = \{0\}$. Suppose $A^T A v = 0$. Then Av is in $\mathrm{Nul}(A^T) = \mathrm{Col}(A)^\perp$. But Av is in $\mathrm{Col}(A)$ as well, so $Av = 0$, and hence $v = 0$ because the columns of A are linearly independent.

Theorem

Let $\{v_1, v_2, \ldots, v_m\}$ be a *linearly independent* set in $\mathbf{R}^n$, and let

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{pmatrix}.$$

Then the $m \times m$ matrix $A^T A$ is invertible.

Let W be a subspace of $\mathbf{R}^n$ and let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be the orthogonal projection with respect to W. Let $\{v_1, v_2, \ldots, v_m\}$ be a *basis* for W and let A be the matrix with columns $v_1, v_2, \ldots, v_m$. To compute $T(x) = x_W$ you solve $A^T Av = Ax$; then $x_W = Av$.

$$v = (A^T A)^{-1}(A^T x) \implies T(x) = Av = \big[A(A^T A)^{-1} A^T\big]x.$$

If the columns of A are a *basis* for W then the matrix for T is

$$A(A^T A)^{-1} A^T.$$

Problem: Let $L = \text{Span}\left\{ \binom{3}{2} \right\}$ and let $T \colon \mathbf{R}^2 \to \mathbf{R}^2$ be the orthogonal projection onto L. Compute the matrix A for T.

The set $\left\{ \binom{3}{2} \right\}$ is a basis for L, so

$$A = u(u^T u)^{-1} u^T = \frac{1}{u \cdot u} u u^T = \frac{1}{13} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \begin{pmatrix} 3 & 2 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 9 & 6 \\ 6 & 4 \end{pmatrix}.$$

Matrix of Projection onto a Line

If $L = \text{Span}\{u\}$ is a line in $\mathbf{R}^n$, then the matrix for projection onto L is

$$\frac{1}{u \cdot u} u u^T.$$

Problem: Let

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x_1 - x_2 + x_3 = 0 \right\}$$

and let $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ be orthogonal projection onto W. Compute the matrix B for T.

In the slides for the last lecture we computed $W = \operatorname{Col} A$ for

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The columns are linearly independent, so they form a basis for W. Hence

$$\begin{aligned} B = A(A^TA)^{-1}A^T = A\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}^{-1} A^T &= \frac{1}{3} A \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} A^T \\ &= \frac{1}{3}\begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}. \end{aligned}$$

Poll

Let W be a subspace of $\mathbf{R}^n$ which is neither the zero subspace nor all of $\mathbf{R}^n$.

Poll

Let A be the matrix for proj_W. What is/are the eigenvalue(s) of A?

A. 0 B. 1 C. -1 D. $0, 1$ E. $1, -1$ F. $0, -1$ G. $-1, 0, 1$

The 1-eigenspace is W.

The 0-eigenspace is $W^\perp$.

We have $\dim W + \dim W^\perp = n$, so that gives n linearly independent eigenvectors already.

So the answer is D.

Theorem

Let W be an m-dimensional subspace of $\mathbf{R}^n$, let $T\colon \mathbf{R}^n \to W$ be the projection, and let A be the matrix for T. Then:

1. $\operatorname{Col} A = W$, which is the 1-eigenspace.
2. $\operatorname{Nul} A = W^\perp$, which is the 0-eigenspace.
3. $A^2 = A$.
4. A is similar to the diagonal matrix with m ones and $n - m$ zeros on the diagonal.

Proof of 4: Let $v_1, v_2, \ldots, v_m$ be a basis for W, and let $v_{m+1}, v_{m+2}, \ldots, v_n$ be a basis for $W^\perp$. These are (linearly independent) eigenvectors with eigenvalues 1 and 0, respectively, and they form a basis for $\mathbf{R}^n$ because there are n of them.

Example: If W is a plane in $\mathbf{R}^3$, then A is similar to projection onto the xy-plane:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A Projection Matrix is Diagonalizable

Let W be an m-dimensional subspace of $\mathbf{R}^n$, let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be the orthogonal projection onto W, and let A be the matrix for T. Here's how to diagonalize A:

- Find a basis $\{v_1, v_2, \ldots, v_m\}$ for W.
- Find a basis $\{v_{m+1}, v_{m+2}, \ldots, v_n\}$ for $W^\perp$.
- Then

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \overbrace{1 \quad 0 \quad 0}^{m \text{ ones, } n-m \text{ zeros}} \\ 0 \quad 1 \quad 0 \\ 0 \quad 0 \quad 0 \end{pmatrix} \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}^{-1}$$

Remark: If you already have a basis for W, then it's faster to compute $A(A^TA)^{-1}A^T$.

Problem: Let

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x_1 - x_2 + x_3 = 0 \right\}$$

and let $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ be orthogonal projection onto W. Compute the matrix B for T.

As we have seen several times, a basis for W is

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

By definition, W is the orthogonal complement of the line spanned by $(1, -1, 1)$, so $W^\perp = \mathrm{Span}\{(1, -1, 1)\}$. Hence

$$B = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \frac{1}{3} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}.$$

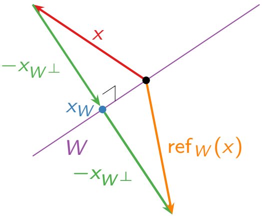
General Reflections (Just for fun!)

Let W be a subspace of $\mathbf{R}^n$ and let x be a vector in $\mathbf{R}^n$.

Definition

The **reflection** of x over W is the vector $\mathrm{ref}_W(x) = x - 2x_{W^\perp}$.

In other words, to find $\mathrm{ref}_W(x)$ one starts at x, then moves to $x - x_{W^\perp} = x_W$, then continues in the same direction one more time, to end on the opposite side of W.



Since $x_{W^\perp} = x - x_W$ we have

$$\mathrm{ref}_W(x) = x - 2(x - x_W) = 2x_W - x.$$

If T is the orthogonal projection, then

$$\mathrm{ref}_W(x) = 2T(x) - x.$$

Theorem

Let W be an m-dimensional subspace of $\mathbf{R}^n$, and let A be the matrix for ref_W. Then

1. $\mathrm{ref}_W \circ \mathrm{ref}_W$ is the identity transformation and A^2 is the identity matrix.
2. ref_W and A are invertible; they are their own inverses.
3. The 1-eigenspace of A is W and the -1-eigenspace of A is $W^\perp$.
4. A is similar to the diagonal matrix with m ones and $n-m$ negative ones on the diagonal.
5. If B is the matrix for the orthogonal projection onto W, then $A = 2B - I_n$.

Example: Let

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x_1 - x_2 + x_3 = 0 \right\}.$$

The matrix for ref_W is

$$A = 2 \cdot \frac{1}{3}\begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} - I_3 = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{pmatrix}.$$

Summary

Today we considered orthogonal projection as a transformation.

- Orthogonal projection is a linear transformation.
- We gave three methods to compute its matrix.
- Four if you count the special case when W is a line.
- The matrix for projection onto W has eigenvalues 1 and 0 with eigenspaces W and $W^\perp$.
- A projection matrix is diagonalizable.
- (Just for fun!) Reflection is $2\times$projection minus the identity.