Math 1553 Conceptual question list §§4.1-5.6

Solutions

Worksheet 8 (4.1-5.1)

1. Let A be an $n \times n$ matrix.

a) If $\det(A) = 1$ and c is a scalar, what is $\det(cA)$?

b) Using cofactor expansion, explain why $\det(A) = 0$ if A has adjacent identical columns.

Solution.

a) By the properties of the determinant, scaling one row by c multiplies the determinant by c. When we take cA for an $n \times n$ matrix A, we are multiplying *each* row by c. This multiplies the determinant by c a total of n times. Thus, if A is $n \times n$ and $\det(A) = 1$, then

$$\det(cA) = c^n \det(A) = c^n(1) = c^n.$$

b) If A has identical adjacent columns, then the cofactor expansions will be identical, except the signs of the cofactors will be opposite (due to the $(-1)^{\text{power}}$ factors).

Therefore, $\det(A) = -\det(A)$, so $\det A = 0$.

2. In this problem, you need not explain your answers; just circle the correct one(s). Let A be an $n \times n$ matrix.

a) Which **one** of the following statements is correct?

1. An eigenvector of A is a vector v such that $Av = \lambda v$ for a nonzero scalar λ.
2. An eigenvector of A is a nonzero vector v such that $Av = \lambda v$ for a scalar λ.
3. An eigenvector of A is a nonzero scalar λ such that $Av = \lambda v$ for some vector v.
4. An eigenvector of A is a nonzero vector v such that $Av = \lambda v$ for a nonzero scalar λ.

b) Which **one** of the following statements is **not** correct?

1. An eigenvalue of A is a scalar λ such that $A - \lambda I$ is not invertible.
2. An eigenvalue of A is a scalar λ such that $(A - \lambda I)v = 0$ has a solution.

3. An eigenvalue of A is a scalar λ such that $Av = \lambda v$ for a nonzero vector v.
4. An eigenvalue of A is a scalar λ such that $\det(A - \lambda I) = 0$.

Solution.

a) Statement 2 is correct: an eigenvector must be nonzero, but its eigenvalue may be zero.

b) Statement 2 is incorrect: the solution v must be nontrivial.

3. True or false: If v_1 and v_2 are linearly independent eigenvectors of an $n \times n$ matrix A, then they must correspond to different eigenvalues.

Solution.

False. For example, if $A = I_2$ then e_1 and e_2 are linearly independent eigenvectors both corresponding to the eigenvalue $\lambda = 1$.

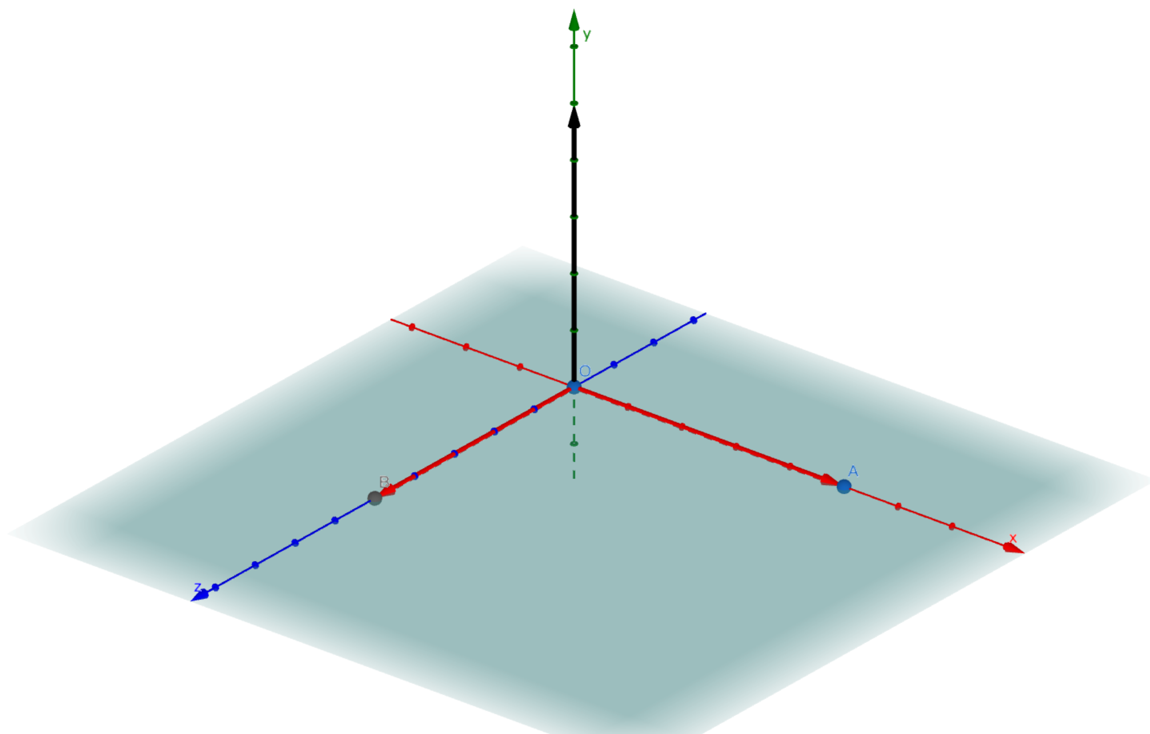
4. In what follows, T is a linear transformation with matrix A. Find the eigenvectors and eigenvalues of A without doing any matrix calculations. (Draw a picture!)

a) $T =$ projection onto the xz-plane in $\mathbf{R}^3$.

b) $T =$ reflection over $y = 2x$ in $\mathbf{R}^2$.

Solution.

a) Here is a picture you can play with https://www.geogebra.org/calculator/sxhzwmxy

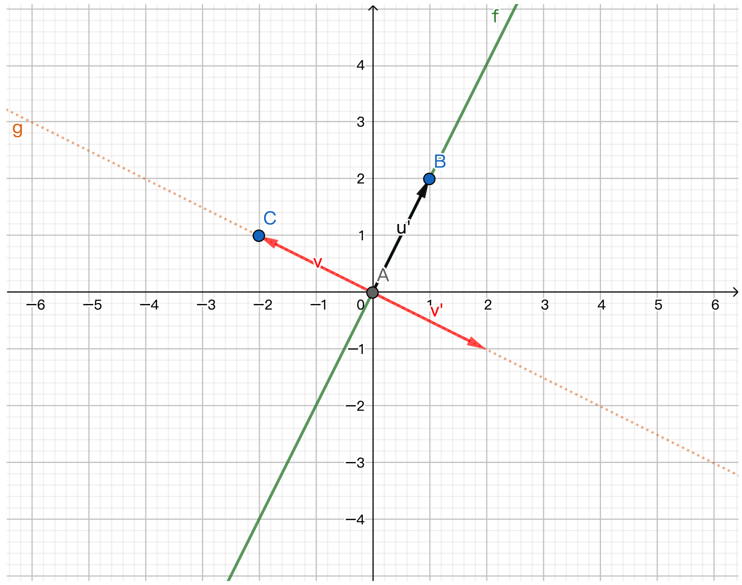


$T(x, y, z) = (x, 0, z)$, so T fixes every vector in the xz-plane and destroys every vector of the form $(0, a, 0)$ with a real. Therefore, $\lambda = 1$ and $\lambda = 0$ are eigenvalues and in fact they are the only eigenvalues since their combined eigenvectors span all of $\mathbf{R}^3$.

The eigenvectors for $\lambda = 1$ are all vectors of the form $\begin{pmatrix} x \\ 0 \\ z \end{pmatrix}$ where at least one of x and z is nonzero, and the eigenvectors for $\lambda = 0$ are all vectors of the form $\begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$ where $y \neq 0$. In other words:

The 1-eigenspace consists of all vectors in $\text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$, while the 0-eigenspace consists of all vectors in $\text{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

b) Here is the picture you can play with `https://www.geogebra.org/calculator/xxmhzgev`



T fixes every vector along the line $y = 2x$, so $\lambda = 1$ is an eigenvalue and its eigenvectors are all vectors $\begin{pmatrix} t \\ 2t \end{pmatrix}$ where $t \neq 0$.

T flips every vector along the line perpendicular to $y = 2x$, which is $y = -\frac{1}{2}x$ (for example, $T(-2, 1) = (2, -1)$). Therefore, $\lambda = -1$ is an eigenvalue and its eigenvectors are all vectors of the form $\begin{pmatrix} s \\ -\frac{1}{2}s \end{pmatrix}$ where $s \neq 0$.

supplemental (4.1-5.1)

1. **a)** Is there a real 2×2 matrix A that satisfies $A^4 = -I_2$? Either write such an A, or show that no such A exists.
(hint: think geometrically! The matrix $-I_2$ represents rotation by π radians).

b) Is there a real 3×3 matrix A that satisfies $A^4 = -I_3$? Either write such an A, or show that no such A exists.

Solution.

a) Yes. Just take A to be the matrix of counterclockwise rotation by $\frac{\pi}{4}$ radians:

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then A^2 gives rotation c.c. by $\frac{\pi}{2}$ radians, A^3 gives rotation c.c. by $\frac{3\pi}{4}$ radians, and A^4 gives rotation c.c. by π radians, which has matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_2$.

b) No. If $A^4 = -I$ then

$$[\det(A)]^4 = \det(A^4) = \det(-I) = (-1)^3 = -1.$$

In other words, if $A^4 = -I$ then $[\det(A)]^4 = -1$, which is impossible since $\det(A)$ is a real number.

Similarly, $A^4 = -I$ is impossible if A is 5×5, 7×7, etc.

2. True or false. Answer true if the statement is always true. Otherwise, answer false.

a) If A and B are $n \times n$ matrices and A is row equivalent to B, then A and B have the same eigenvalues.

b) If A is an $n \times n$ matrix and its eigenvectors form a basis for $\mathbf{R}^n$, then A is invertible.

c) If 0 is an eigenvalue of the $n \times n$ matrix A, then $\operatorname{rank}(A) < n$.

d) The diagonal entries of an $n \times n$ matrix A are its eigenvalues.

e) If A is invertible and 2 is an eigenvalue of A, then $\frac{1}{2}$ is an eigenvalue of A^{-1}.

f) If $\det(A) = 0$, then 0 is an eigenvalue of A.

g) If v and w are eigenvectors of a square matrix A, then so is $v + w$.

Solution.

a) False. For instance, the matrices $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are row equivalent, but have different eigenvalues.

b) False. For example, the zero matrix is not invertible but its eigenvectors form a basis for $\mathbf{R}^n$.

c) True. If $\lambda = 0$ is an eigenvalue of A then A is not invertible so its associated transformation $T(x) = Ax$ is not onto, hence rank$(A) < n$.

d) False. This is true if A is triangular, but not in general.

For example, if $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ then the diagonal entries are 2 and 0 but the only eigenvalue is $\lambda = 1$, since solving the characteristic equation gives us

$$(2-\lambda)(-\lambda) - (1)(-1) = 0 \qquad \lambda^2 - 2\lambda + 1 = 0 \qquad (\lambda - 1)^2 = 0 \qquad \lambda = 1.$$

e) True. Let v be an eigenvector corresponding to the eigenvalue 2.

$$Av = 2v \implies A^{-1}Av = A^{-1}(2v) \implies v = 2A^{-1}v \implies \frac{1}{2}v = A^{-1}v.$$

Therefore, v is an eigenvector of A^{-1} corresponding to the eigenvalue $\frac{1}{2}$.

f) True. If $\det(A) = 0$ then A is not invertible, so $Av = 0v$ has a nontrivial solution.

g) False. Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Note e_1 is an eigenvector corresponding to $\lambda = 1$ and e_2 is an eigenvector corresponding to $\lambda = 2$, but $e_1 + e_2$ is not an eigenvalue of A.

Worksheet 9 (5.1-5.4)

1. True or false. If the statement is always true, answer true and justify why it is true. Otherwise, answer false and give an example that shows it is false. In every case, assume that A is an $n \times n$ matrix.

a) The entries on the main diagonal of A are the eigenvalues of A.

b) The number λ is an eigenvalue of A if and only if there is a nonzero solution to the equation $(A - \lambda I)x = 0$.

c) To find the eigenvectors of A, we reduce the matrix A to row echelon form.

d) If A is invertible and 2 is an eigenvalue of A, then $\frac{1}{2}$ is an eigenvalue of A^{-1}.

e) If Nul(A) has dimension at least 1, then Nul(A) is the eigenspace of A corresponding to the eigenvalue 0.

Solution.

a) False. This is true if A is triangular, but not in general.

For example, if $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ then the diagonal entries are 2 and 0 but the only eigenvalue is $\lambda = 1$, since solving the characteristic equation gives us

$$(2-\lambda)(-\lambda) - (1)(-1) = 0 \qquad \lambda^2 - 2\lambda + 1 = 0 \qquad (\lambda - 1)^2 = 0 \qquad \lambda = 1.$$

b) True.

$$(A - \lambda I)x = 0 \iff Ax - \lambda x = 0 \iff Ax = \lambda x.$$

Therefore, $(A - \lambda I)x = 0$ has a nonzero solution if and only if $Ax = \lambda x$ has a nonzero solution, which is to say that λ is an eigenvalue of A.

c) False. The RREF of A will only compute the eigenvectors with eigenvalue zero, or will tell us that zero is not an eigenvalue. To get the eigenvectors corresponding to an eigenvalue λ, we put $A - \lambda I$ into RREF and write the solutions of $\left(A - \lambda I \mid 0\right)$ in parametric vector form.

d) True. Let v be an eigenvector corresponding to the eigenvalue 2.

$$Av = 2v \implies A^{-1}Av = A^{-1}(2v) \implies v = 2A^{-1}v \implies \frac{1}{2}v = A^{-1}v.$$

Therefore, v is an eigenvector of A^{-1} corresponding to the eigenvalue $\frac{1}{2}$.

e) True. For every v in Nul A, we have $Av = 0v$. If $v \neq 0$, this is exactly the definition of v being an eigenvector corresponding to the eigenvalue 0. If Nul$A$ has dimension at least 1, then infinitely many nonzero vectors satisfy $Av = 0$, so 0 is an eigenvalue of A (and every nonzero vector v satisfying $Av = 0$ is an eigenvector of A) and Nul A is the 0-eigenspace of A.

2. Suppose A is an $n \times n$ matrix satisfying $A^2 = 0$. Find all eigenvalues of A. Justify your answer.

Solution.

If λ is an eigenvalue of A and $v \neq 0$ is a corresponding eigenvector, then

$$Av = \lambda v \implies A(Av) = A\lambda v \implies A^2 v = \lambda(Av) \implies 0 = \lambda(\lambda v) \implies 0 = \lambda^2 v.$$

Since $v \neq 0$ this means $\lambda^2 = 0$, so $\lambda = 0$. This shows that 0 is the only possible eigenvalue of A.

On the other hand, $\det(A) = 0$ since $(\det(A))^2 = \det(A^2) = \det(0) = 0$, so 0 must be an eigenvalue of A. Therefore, the only eigenvalue of A is 0.

3. Answer yes, no, or maybe. Justify your answers. In each case, A is a matrix whose entries are real numbers.

a) Suppose $A = \begin{pmatrix} 3 & 0 & 0 \\ 5 & 1 & 0 \\ -10 & 4 & 7 \end{pmatrix}$. Then the characteristic polynomial of A is

$$\det(A - \lambda I) = (3 - \lambda)(1 - \lambda)(7 - \lambda).$$

b) If A is a 3×3 matrix with characteristic polynomial $-\lambda(\lambda - 5)^2$, then the 5-eigenspace is 2-dimensional.

c) If A is an invertible 2×2 matrix, then A is diagonalizable.

Solution.

a) Yes. Since $A - \lambda I$ is triangular, its determinant is the product of its diagonal entries.

b) Maybe. The geometric multiplicity of $\lambda = 5$ can be 1 or 2. For example, the matrix $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has a 5-eigenspace which is 2-dimensional, whereas the matrix $\begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has a 5-eigenspace which is 1-dimensional. Both matrices have characteristic polynomial $-\lambda(5 - \lambda)^2$.

c) Maybe. The identity matrix is invertible and diagonalizable, but the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is invertible but not diagonalizable.

supplemental (5.1-5.4)

1. True or false. If the statement is always true, answer true and justify why it is true. Otherwise, answer false and give an example that shows it is false.

a) If A and B are $n \times n$ matrices with the same eigenvectors, then A and B have the same characteristic polynomial.

b) If A is a 3×3 matrix with characteristic polynomial $-\lambda^3 + \lambda^2 + \lambda$, then A is invertible.

Solution.

a) False: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ have the same eigenvectors (all nonzero vectors in $\mathbf{R}^2$) but characteristic polynomials λ^2 and $(1-\lambda)^2$, respectively.

b) False: $\lambda = 0$ is a root of the characteristic polynomial, so 0 is an eigenvalue, and A is not invertible.

2. True or false. Answer true if the statement is always true. Otherwise, answer false.

a) If A is an invertible matrix and A is diagonalizable, then A^{-1} is diagonalizable.

b) A diagonalizable $n \times n$ matrix admits n linearly independent eigenvectors.

c) If A is diagonalizable, then A has n distinct eigenvalues.

Solution.

a) True. If $A = PDP^{-1}$ and A is invertible then its eigenvalues are all nonzero, so the diagonal entries of D are nonzero and thus D is invertible (pivot in every diagonal position). Thus, $A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1}D^{-1}P^{-1} = PD^{-1}P^{-1}$.

b) True. By the Diagonalization Theorem, an $n \times n$ matrix is diagonalizable *if and only if* it admits n linearly independent eigenvectors.

c) False. For instance, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is diagonal but has only one eigenvalue.

3. Give examples of 2×2 matrices with the following properties. Justify your answers.

a) A matrix A which is invertible and diagonalizable.

b) A matrix B which is invertible but not diagonalizable.

c) A matrix C which is not invertible but is diagonalizable.

d) A matrix D which is neither invertible nor diagonalizable.

Solution.

a) We can take any diagonal matrix with nonzero diagonal entries:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

b) A shear has only one eigenvalue $\lambda = 1$. The associated eigenspace is the x-axis, so there do not exist two linearly independent eigenvectors. Hence it is not diagonalizable.

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

c) We can take any diagonal matrix with some zero diagonal entries:

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

d) Such a matrix can only have the eigenvalue zero — otherwise it would have two eigenvalues, hence be diagonalizable. Thus the characteristic polynomial is $f(\lambda) = \lambda^2$. Here is a matrix with trace and determinant zero, whose zero-eigenspace (i.e., null space) is not all of $\mathbf{R}^2$:

$$D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Worksheet 10 (5.4-5.6)

1. True or false. If the statement is always true, answer true and justify why it is true. Otherwise, answer false and give an example that shows it is false. If not explicitly stated, assume A, B are $n \times n$ matrices.

a) If A is diagonalizable and B is row equivalent to A, then B is also diagonalizable.

b) If A and B are diagonalizable, then AB is diagonalizable.

c) A 3×3 matrix A can have a non-real complex eigenvalue with multiplicity 2.

d) If A is the 3×3 the matrix for the orthogonal projection of vectors in $\mathbf{R}^3$ onto the plane $x + y + z = 0$, then A is diagonalizable.

Solution.

a) No, for example, $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. B is not diagonalizable.

b) No, for example, $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, and $B = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$. $AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

c) No. If c is a (non-real) complex eigenvalue with multiplicity 2, then its conjugate $\overline{c}$ is an eigenvalue with multiplicity 2 since complex eigenvalues always occur in conjugate pairs. This would mean A has a characteristic polynomial of degree 4 or more, which is impossible since A is 3×3.

d) Yes, it is diagonalizable. Since we can clearly find three independent eigenvectors. For $\lambda_1 = 0$, $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. For $\lambda_2 = \lambda_3 = 1$, $v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

supplemental (5.4-5.6)

1. True/False

a) If A is the matrix that implements rotation by $143°$ in $\mathbf{R}^2$, then A has no real eigenvalues.

b) A 3×3 matrix can have eigenvalues $3, 5$, and $2+i$.

c) If $v = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = 1-i$, then $w = \begin{pmatrix} 2i-1 \\ i \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = 1-i$.

Solution.

a) True. If A had a real eigenvalue λ, then we would have $Ax = \lambda x$ for some nonzero vector x in $\mathbf{R}^2$. This means that x would lie on the same line through the origin as the rotation of x by $143°$, which is impossible.

b) False. If $2+i$ is an eigenvalue then so is its conjugate $2-i$.

c) True. Any nonzero complex multiple of v is also an eigenvector for eigenvalue $1-i$, and $w = iv$.