Supplemental problems: Chapter 4, Determinants

1. If $A$ is an $n \times n$ matrix, is it necessarily true that $\det(-A) = -\det(A)$? Justify your answer.

2. Let $A$ be an $n \times n$ matrix.
   a) Using cofactor expansion, explain why $\det(A) = 0$ if $A$ has a row or a column of zeros.
   b) Using cofactor expansion, explain why $\det(A) = 0$ if $A$ has adjacent identical columns.

3. Find the volume of the parallelepiped in $\mathbb{R}^4$ naturally determined by the vectors
   \[
   \begin{bmatrix}
   4 \\
   1 \\
   3 \\
   8
   \end{bmatrix}, \quad
   \begin{bmatrix}
   0 \\
   7 \\
   0 \\
   3
   \end{bmatrix}, \quad
   \begin{bmatrix}
   0 \\
   2 \\
   1 \\
   1
   \end{bmatrix}, \quad
   \begin{bmatrix}
   5 \\
   -5 \\
   0 \\
   7
   \end{bmatrix}.
   \]

4. Let $A = \begin{pmatrix} -1 & 1 \\ 1 & 7 \end{pmatrix}$, and define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = Ax$. Find the area of $T(S)$, if $S$ is a triangle in $\mathbb{R}^2$ with area 2.

5. Let
   \[
   A = \begin{pmatrix}
   7 & 1 & 4 & 1 \\
   -1 & 0 & 0 & 6 \\
   9 & 0 & 2 & 3 \\
   0 & 0 & 0 & -1
   \end{pmatrix} \quad \text{and} \quad
   B = \begin{pmatrix}
   0 & 1 & 5 & 4 \\
   1 & -1 & -3 & 0 \\
   -1 & 0 & 5 & 4 \\
   3 & -3 & -2 & 5
   \end{pmatrix}
   \]
   a) Compute $\det(A)$.
   b) Compute $\det(B)$.
   c) Compute $\det(AB)$.
   d) Compute $\det(A^2B^{-1}AB^2)$.

6. If $A$ is a $3 \times 3$ matrix and $\det(A) = 1$, what is $\det(-2A)$?

7. a) Is there a real $2 \times 2$ matrix $A$ that satisfies $A^4 = -I_2$? Either write such an $A$, or show that no such $A$ exists.
   (hint: think geometrically! The matrix $-I_2$ represents rotation by $\pi$ radians).
   b) Is there a real $3 \times 3$ matrix $A$ that satisfies $A^4 = -I_3$? Either write such an $A$, or show that no such $A$ exists.