Supplemental problems: §3.1

1. Review from 2.6-2.9. Fill in the blanks: If A is a 7×6 matrix and the solution set for $Ax=0$ is a plane, then the column space of A is a ___4___ -dimensional subspace of $\mathbf{R}^{\boxed{7}}$.
Reason: $\operatorname{rank}(A)+\operatorname{nullity}(A)=6 \qquad \operatorname{rank}(A)+2=6 \qquad \operatorname{rank}(A)=4$

2. Review from 2.6-2.9: Consider the matrix A below and its RREF:

$$A=\begin{pmatrix} 1 & 2 & -1 & -1 \\ -2 & -4 & -6 & 2 \\ 1 & 2 & -5 & -1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) Write a basis for Col A.

The pivot columns (1 and 3) form a basis for Col(A), but really column 3 and any other column will work.

$$\left\{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -6 \\ -5 \end{pmatrix}\right\}.$$

b) Find a basis for Nul A. From the RREF of A, we see the solution set is

$$x_1+2x_2-x_4=0, \quad x_3=0,$$

so $x_1=-2x_2+x_4$, x_2 and x_4 are free, and $x_3=0$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2x_2+x_4 \\ x_2 \\ 0 \\ x_4 \end{pmatrix} = x_2\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \qquad \text{A basis is} \quad \left\{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

c) Is there a matrix B so that Col(B) = Nul(A)? If yes, write such a B. If not, justify why no such matrix B exists.

Yes. Just take the columns of B to be a set whose span is Nul A, for example

$$B=\begin{pmatrix} -2 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

3. Suppose T is a matrix transformation and the range of T is the subspace

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \,\middle|\, x - 3y + 4z = 0 \right\}$$

of $\mathbf{R}^3$, which contains the vectors $v_1 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$. Is $\{v_1, v_2\}$ a basis for the range of T?

Solution.

Yes. We know that V is a 2-dimensional subspace of $\mathbf{R}^3$ since $V = \mathrm{Nul}\begin{pmatrix} 1 & -3 & 4 \end{pmatrix}$ which corresponds to a homogeneous system with two free variables. Since $\{v_1, v_2\}$ is clearly a linearly independent set in V and $\dim(V) = 2$, it forms a basis for V by the Basis Theorem.

4. True or false. If the statement is *always* true, answer TRUE. Otherwise, circle FALSE.

a) The matrix transformation $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ performs reflection across the x-axis in $\mathbf{R}^2$. TRUE $\boxed{\text{FALSE}}$ (T reflects across the y-axis then projects onto the x-axis)

b) The matrix transformation $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ performs rotation counter-clockwise by 90° in $\mathbf{R}^2$. TRUE $\boxed{\text{FALSE}}$ (T rotates clockwise 90°)

5. Let T be the matrix transformation $T(x) = Ax$, where $A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ -1 & 0 & -1 & -2 \\ 2 & 2 & 4 & 2 \end{pmatrix}$.
What is the domain of T? What is its codomain? Find a basis for the range of T and a basis for the kernel of T (the kernel of T is the set of all vectors satisfying $T(x) = 0$).

Solution: The domain of T is $\mathbf{R}^4$ and the codomain of T is $\mathbf{R}^3$. The range of T is Col A and the kernel of T is Nul A. We row-reduce $\left(A \mid 0 \right)$:

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ -1 & 0 & -1 & -2 & 0 \\ 2 & 2 & 4 & 2 & 0 \end{array}\right) \xrightarrow[R_3 = R_3 - 2R_1]{R_2 = R_2 + R_1} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 = R_1 - R_2} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

We see x_3 and x_4 are free, and $x_1 = -x_3 - 2x_4$ and $x_2 = -x_3 + x_4$. The parametric vector form for elements of Nul A is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_3 - 2x_4 \\ -x_3 + x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$ A basis for kernel(T) is $\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

A basis for Range(T) is given by the pivot columns of A, namely $\left\{\begin{pmatrix}1\\-1\\2\end{pmatrix},\begin{pmatrix}1\\0\\2\end{pmatrix}\right\}$.

In this case, any two columns of A will actually form a basis for Col$A$, so any two columns of A will be a correct answer.

6. The matrix $A=\begin{pmatrix}1&2&1\\1&2&1\\1&0&0\end{pmatrix}$ has RREF $\begin{pmatrix}1&0&0\\0&1&\frac{1}{2}\\0&0&0\end{pmatrix}$. Define a matrix transformation by $T(x)=Ax$. Is $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$ a basis for the range of T?

Solution.

No. The range of T is Col A. To get a basis for Col A, we use the The pivot columns of the original matrix A, not its RREF.

7. In each case, a matrix is given below.

Match each matrix to its corresponding transformation (choosing from (i) through (viii)) by writing that roman numeral next to the matrix. Note there are four matrices and eight options, so not every option is used.

$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ This is (i) Reflection across x-axis

$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ This is (v) Rotation counterclockwise by $\pi/2$ radians

$\begin{pmatrix}0&0\\0&1\end{pmatrix}$ This is (viii) Projection onto the y-axis

$\begin{pmatrix}2&0\\0&2\end{pmatrix}$ This is (iii) Scaling by a factor of 2

(i) Reflection across x-axis
(ii) Reflection across y-axis
(iii) Scaling by a factor of 2
(iv) Scaling by a factor of 1/2
(v) Rotation counterclockwise by $\pi/2$ radians
(vi) Rotation clockwise by $\pi/2$ radians
(vii) Projection onto the x-axis
(viii) Projection onto the y-axis

Supplemental problems: §3.2

1. Let A be a 3×4 matrix with column vectors v_1, v_2, v_3, v_4, and suppose $v_2 = 2v_1 - 3v_4$. Consider the matrix transformation $T(x) = Ax$.

a) Is it possible that T is one-to-one? If yes, justify why. If no, find distinct vectors v and w so that $T(v) = T(w)$.

b) Is it possible that T is onto? Justify your answer.

Solution.

a) From the linear dependence condition we were given, we get

$$-2v_1 + v_2 + 3v_4 = 0.$$

The corresponding vector equation is just

$$\begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{so} \quad A \begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, $v = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix}$ and $w = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ both satisfy $Av = Aw = 0$, so T cannot be one-to-one.

b) Yes. If $\{v_1, v_3, v_4\}$ is linearly independent then A will have a pivot in every row and T will be onto. Such a matrix A is

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{pmatrix}.$$

2. **a)** Which of the following are onto transformations? (Check all that apply.)

- [x] $T : \mathbf{R}^3 \to \mathbf{R}^3$, reflection over the xy-plane
- [ ] $T : \mathbf{R}^3 \to \mathbf{R}^3$, projection onto the xy-plane
- [x] $T : \mathbf{R}^3 \to \mathbf{R}^2$, project onto the xy-plane, forget the z-coordinate
- [x] $T : \mathbf{R}^2 \to \mathbf{R}^2$, scale the x-direction by 2

b) Let A be a square matrix and let $T(x) = Ax$. Which of the following guarantee that T is onto? (Check all that apply.)

- [x] T is one-to-one

☐ $Ax=0$ is consistent

3. Find all real numbers h so that the transformation $T:\mathbf{R}^3\to\mathbf{R}^2$ given by

$$T(v)=\begin{pmatrix}-1 & 0 & 2-h\\ h & 0 & 3\end{pmatrix}v$$

is onto.

Solution.

We row-reduce A to find when it will have a pivot in every row:

$$\begin{pmatrix}-1 & 0 & 2-h\\ h & 0 & 3\end{pmatrix}\xrightarrow{R_2=R_2+hR_1}\begin{pmatrix}-1 & 0 & 2-h\\ 0 & 0 & 3+h(2-h)\end{pmatrix}.$$

The matrix has a pivot in every row unless

$$3+h(2-h)=0,\qquad h^2-2h-3=0,\qquad (h-3)(h+1)=0.$$

Therefore, T is onto as long as $h\neq 3$ and $h\neq -1$.

4. Let $T:\mathbf{R}^n\to\mathbf{R}^m$ be a matrix transformation $T(x)=Ax$. Which of the following conditions guarantee that T *must* be one-to-one? Circle all that apply.
(i) A has m pivots.
(ii) The columns of A are linearly independent.
(iii) For each input vector x in $\mathbf{R}^n$, there is exactly one output vector $T(x)$ in $\mathbf{R}^m$.
(iv) The equation $Ax=b$ has exactly one solution for each b in $\mathbf{R}^m$.

Solution.

The correct answers are (ii) and (iv).
(ii) is equivalent to T being one-to-one, and (iv) guarantees T is one-to-one and onto. However, (i) is not necessarily true and (iii) is just the definition of a function.

5. Answer each question.

a) Suppose $S:\mathbf{R}^3\to\mathbf{R}^2$ is the matrix transformation $S(x)=\begin{pmatrix}1 & 0 & 2\\ 0 & 1 & 3\end{pmatrix}x$.
Is S one-to-one? NO

Is S onto? YES

b) Suppose $T:\mathbf{R}^2\to\mathbf{R}^2$ is given by $T(x,y)=(x-y,\ x-y)$.
Is T one-to-one? NO

Is T onto? NO

c) Suppose $T : \mathbf{R}^n \to \mathbf{R}^m$ is a one-to-one matrix transformation. Which one of the following *must* be true? (cicle one)

$$m \geq n$$

6. Which of the following transformations are onto?
Circle all that apply.

a) $T : \mathbf{R}^2 \to \mathbf{R}^2$ that rotates counterclockwise by $\dfrac{\pi}{12}$ radians.

b) The transformation $T(x) = Ax$, where A is a 4×3 matrix with three pivots.

c) $T : \mathbf{R}^3 \to \mathbf{R}^3$ that reflects across the yz-plane.

Solution.

The transformations (a) and (c) are onto. Note that (b) is not onto since A doesn't have a pivot in every row. In (b), range(T) is a 3-dimensional subspace of $\mathbf{R}^4$.