Section 2.6

Subspaces
Today we will discuss **subspaces** of $\mathbb{R}^n$.

A subspace turns out to be the same as a span, except we don’t know *which* vectors it’s the span of.

This arises naturally when you have, say, a plane through the origin in $\mathbb{R}^3$ which is *not* defined (a priori) as a span, but you still want to say something about it.

$$x + 3y + z = 0$$
Definition of Subspace

Definition
A **subspace** of $\mathbb{R}^n$ is a subset $V$ of $\mathbb{R}^n$ satisfying:

1. The zero vector is in $V$. **“not empty”**
2. If $u$ and $v$ are in $V$, then $u + v$ is also in $V$. **“closed under addition”**
3. If $u$ is in $V$ and $c$ is in $\mathbb{R}$, then $cu$ is in $V$. **“closed under $\times$ scalars”**

Fast-forward

Every subspace is a span, and every span is a subspace.

A subspace is a span of some vectors, but you haven’t computed what those vectors are yet.
Definition of Subspace

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2. If $u$ and $v$ are in $V$, then $u + v$ is also in $V$. “closed under addition”
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What does this mean?

- If $v$ is in $V$, then all scalar multiples of $v$ are in $V$ by (3). In other words, the line through any nonzero vector in $V$ is also in $V$.
- If $u, v$ are in $V$, then $cu$ and $dv$ are in $V$ for any scalars $c, d$ by (3). So $cu + dv$ is in $V$ by (2). So $\text{Span}\{u, v\}$ is contained in $V$.
- Likewise, if $v_1, v_2, \ldots, v_n$ are all in $V$, then $\text{Span}\{v_1, v_2, \ldots, v_n\}$ is contained in $V$: a subspace contains the span of any set of vectors in it.

If you pick enough vectors in $V$, eventually their span will fill up $V$, so:

A subspace is a span of some set of vectors in it.
Examples

Example
A line $L$ through the origin is a subspace: $L$ contains zero and is easily seen to be closed under addition and scalar multiplication.

Example
A plane $P$ through the origin is a subspace: $P$ contains zero; the sum of two vectors in $P$ is also in $P$; and any scalar multiple of a vector in $P$ is also in $P$.

Example
All of $\mathbb{R}^n$: this contains 0, and is closed under addition and scalar multiplication.

Example
The subset $\{0\}$: this subspace contains only one vector.

Note these are all pictures of spans! (Line, plane, space, etc.)
A **subset** of \( \mathbb{R}^n \) is any collection of vectors in \( \mathbb{R}^n \) whatsoever. For example, the unit circle

\[
C = \{(x, y) \text{ in } \mathbb{R}^2 \mid x^2 + y^2 = 1\}
\]

is a subset of \( \mathbb{R}^2 \), but it is not a subspace.

All of the following non-examples on the next slide are still subsets.

A **subspace** is a special kind of subset, that satisfies the three defining properties.
**Non-Example**

A line $L$ (or any other set) that doesn’t contain the origin is not a subspace. Fails: **1**.

**Non-Example**

A circle $C$ is not a subspace. Fails: **1,2,3**. Think: a circle isn't a “linear space.”

**Non-Example**

The first quadrant in $\mathbb{R}^2$ is not a subspace. Fails: **3** only.

**Non-Example**

A line union a plane in $\mathbb{R}^3$ is not a subspace. Fails: **2** only.
Subspaces are Spans, and Spans are Subspaces

Theorem
Any Span \{v_1, v_2, \ldots, v_p\} is a subspace.

Every subspace is a span, and every span is a subspace.

Definition
If \( V = \text{Span}\{v_1, v_2, \ldots, v_p\} \), we say that \( V \) is the subspace generated by or spanned by the vectors \( v_1, v_2, \ldots, v_p \). We call \( \{v_1, v_2, \ldots, v_p\} \) a spanning set for \( V \).

Check:
1. \( 0 = 0v_1 + 0v_2 + \cdots + 0v_p \) is in the span.
2. If, say, \( u = 3v_1 + 4v_2 \) and \( v = -v_1 - 2v_2 \), then
   \[
   u + v = 3v_1 + 4v_2 - v_1 - 2v_2 = 2v_1 + 2v_2
   \]
   is also in the span.
3. Similarly, if \( u \) is in the span, then so is \( cu \) for any scalar \( c \).
Which of the following are subspaces?

A. The empty set \{\}.
B. The solution set to a homogeneous system of linear equations.
C. The solution set to an inhomogeneous system of linear equations.
D. The set of all vectors in \( \mathbb{R}^n \) with rational (fraction) coordinates.

For the ones which are not subspaces, which property(ies) do they not satisfy?

A. This is not a subspace: it does not contain the zero vector.
B. This is a subspace: the solution set is a span, produced by finding the parametric vector form of the solution.
C. This is not a subspace: it does not contain 0.
D. This is not a subspace: it is not closed under multiplication by scalars (e.g. by \( \pi \)).
Subspaces
Verification

Let \( V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \mid ab = 0 \right\} \). Let’s check if \( V \) is a subspace or not.

1. Does \( V \) contain the zero vector? \( \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies ab = 0 \) ✔

3. Is \( V \) closed under scalar multiplication?
   - Let \( \begin{pmatrix} a \\ b \end{pmatrix} \) be (an unknown vector) in \( V \).
   - This means: \( a \) and \( b \) are numbers such that \( ab = 0 \).
   - Let \( c \) be a scalar. Is \( c\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ca \\ cb \end{pmatrix} \) in \( V \)?
   - This means: \((ca)(cb) = 0\).
   - Well, \((ca)(cb) = c^2(ab) = c^2(0) = 0\) ✔

2. Is \( V \) closed under addition?
   - Let \( \begin{pmatrix} a \\ b \end{pmatrix} \) and \( \begin{pmatrix} a' \\ b' \end{pmatrix} \) be (unknown vectors) in \( V \).
   - This means: \( ab = 0 \) and \( a'b' = 0 \).
   - Is \( \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} a+a' \\ b+b' \end{pmatrix} \) in \( V \)?
   - This means: \((a+a')(b+b') = 0\).
   - This is not true for all such \( a, a', b, b' \): for instance, \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) are in \( V \), but their sum \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) is not in \( V \), because \( 1 \cdot 1 \neq 0 \). ❌

We conclude that \( V \) is not a subspace. A picture is above. (It doesn’t look like a span.)
Column Space and Null Space

An $m \times n$ matrix $A$ naturally gives rise to two subspaces.

Definition

- The **column space** of $A$ is the subspace of $\mathbb{R}^m$ spanned by the columns of $A$. It is written $\text{Col } A$.

- The **null space** of $A$ is the set of all solutions of the homogeneous equation $Ax = 0$:
  \[ \text{Nul } A = \{ x \in \mathbb{R}^n \mid Ax = 0 \} . \]
  This is a subspace of $\mathbb{R}^n$.

The column space is defined as a span, so we know it is a subspace.

**Check** that the null space is a subspace:

1. 0 is in Nul $A$ because $A0 = 0$.

2. If $u$ and $v$ are in Nul $A$, then $Au = 0$ and $Av = 0$. Hence
   \[ A(u + v) = Au + Av = 0 , \]
   so $u + v$ is in Nul $A$.

3. If $u$ is in Nul $A$, then $Au = 0$. For any scalar $c$, $A(cu) = cAu = 0$. So $cu$ is in Nul $A$. 
Column Space and Null Space

Example

Let \( A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \).

Let’s compute the column space:

\[
\text{Col} \ A = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.
\]

This is a line in \( \mathbb{R}^3 \).

Let’s compute the null space:

The reduced row echelon form of \( A \) is \( \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \).

This gives the equation \( x + y = 0 \), or

\[
x = -y \quad \text{parametric vector form} \quad \begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \end{pmatrix}.
\]

Hence the null space is \( \text{Span}\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\} \), a line in \( \mathbb{R}^2 \).
The Null Space is a Span

The column space of a matrix $A$ is defined to be a span (of the columns).

The null space is defined to be the solution set to $Ax = 0$. It is a subspace, so it is a span.

**Question**

How to find vectors that span the null space?

**Answer:** Parametric vector form! We know that the solution set to $Ax = 0$ has a parametric form that looks like

$$
\begin{pmatrix}
1 \\
2 \\
1 \\
0
\end{pmatrix} + \lambda \begin{pmatrix}
-2 \\
3 \\
0 \\
1
\end{pmatrix}
$$

if, say, $x_3$ and $x_4$ are the free variables. So

$$\text{Nul } A = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Refer back to the slides for §2.4 (Solution Sets).

**Note:** It is much easier to define the null space first as a subspace, then find spanning vectors *later*, if we need them. This is one reason subspaces are so useful.
A **subspace** is the same as a span of some number of vectors, but we haven’t computed the vectors yet.

To any matrix is associated two subspaces, the **column space** and the **null space**:

- Col $A = \text{the span of the columns of } A$
- Nul $A = \text{the solution set of } Ax = 0$.

**How do you check if a subset is a subspace?**

- Is it a span? Can it be written as a span?
- Can it be written as the column space of a matrix?
- Can it be written as the null space of a matrix?
- Is it all of $\mathbb{R}^n$ or the zero subspace $\{0\}$?
- Can it be written as a type of subspace that we’ll learn about later (eigenspaces, ...)?

If so, then it’s automatically a subspace.

If all else fails:

- Can you verify directly that it satisfies the three defining properties?