Math 1553 Conceptual question list §§2.6-3.6

Worksheet 5 (2.6-3.2)

1. Circle TRUE if the statement is always true, and circle FALSE otherwise.
   a) If \( A \) is a \( 3 \times 10 \) matrix with 2 pivots in its RREF, then \( \dim(\text{Nul}A) = 8 \) and \( \text{rank}(A) = 2 \).
      
      TRUE    FALSE

   b) If \( A \) is an \( m \times n \) matrix and \( Ax = 0 \) has only the trivial solution, then the transformation \( T(x) = Ax \) is onto.
      
      TRUE    FALSE

   c) If \( \{a, b, c\} \) is a basis of a linear space \( V \), then \( \{a, a + b, b + c\} \) is a basis of \( V \) as well.
      
      TRUE    FALSE

2. Write a matrix \( A \) so that \( \text{Col}(A) \) is the solid blue line and \( \text{Nul}(A) \) is the dotted red line drawn below.
supplemental (2.6-3.2)

1. Circle **TRUE** if the statement is always true, and circle **FALSE** otherwise.
   a) If $A$ is a $3 \times 100$ matrix of rank 2, then $\dim(\operatorname{Nul}A) = 97$.
      TRUE  FALSE
   b) If $A$ is an $m \times n$ matrix and $Ax = 0$ has only the trivial solution, then the columns of $A$ form a basis for $\mathbb{R}^m$.
      TRUE  FALSE
   c) The set $V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \mid x - 4z = 0 \right\}$ is a subspace of $\mathbb{R}^4$.
      TRUE  FALSE

2. Circle **T** if the statement is always true, and circle **F** otherwise. You do not need to explain your answer.
   a) If $\{v_1, v_2, v_3, v_4\}$ is a basis for a subspace $V$ of $\mathbb{R}^n$, then $\{v_1, v_2, v_3\}$ is a linearly independent set.
   b) The solution set of a consistent matrix equation $Ax = b$ is a subspace.
   c) A translate of a span is a subspace.

3. True or false (justify your answer). Answer true if the statement is *always* true. Otherwise, answer false.
   a) There exists a $3 \times 5$ matrix with rank 4.
   b) If $A$ is an $9 \times 4$ matrix with a pivot in each column, then
      $$\operatorname{Nul}A = \{0\}.$$ 
   c) There exists a $4 \times 7$ matrix $A$ such that nullity $A = 5$.
   d) If $\{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbb{R}^4$, then $n = 4$.

4. a) True or false: If $A$ is an $m \times n$ matrix and $\operatorname{Nul}(A) = \mathbb{R}^n$, then $\operatorname{Col}(A) = \{0\}$.
   b) Give an example of a $2 \times 2$ matrix whose column space is the same as its null space.
   c) True or false: For some $m$, we can find an $m \times 10$ matrix $A$ whose column span has dimension 4 and whose solution set for $Ax = 0$ has dimension 5.

5. Fill in the blanks: If $A$ is a $7 \times 6$ matrix and the solution set for $Ax = 0$ is a plane, then the column space of $A$ is a _______-dimensional subspace of $\mathbb{R}^\square$. 
6. True or false. If the statement is always true, answer TRUE. Otherwise, circle FALSE.
   a) The matrix transformation \( T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \) performs reflection across the \( x \)-axis in \( \mathbb{R}^2 \). TRUE FALSE
   b) The matrix transformation \( T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \) performs rotation counter-clockwise by 90° in \( \mathbb{R}^2 \). TRUE FALSE

7. Let \( A \) be a \( 3 \times 4 \) matrix with column vectors \( v_1, v_2, v_3, v_4 \), and suppose \( v_2 = 2v_1 - 3v_4 \). Consider the matrix transformation \( T(x) = Ax \).
   a) Is it possible that \( T \) is one-to-one? If yes, justify why. If no, find distinct vectors \( v \) and \( w \) so that \( T(v) = T(w) \).
   b) Is it possible that \( T \) is onto? Justify your answer.

8. Answer each question.
   a) Suppose \( S : \mathbb{R}^3 \to \mathbb{R}^2 \) is the matrix transformation \( S(x) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix} x \).
      Is \( S \) one-to-one? YES NO
      Is \( S \) onto? YES NO
   b) Suppose \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) is given by \( T(x, y) = (x - y, x - y) \).
      Is \( T \) one-to-one? YES NO
      Is \( T \) onto? YES NO
   c) Suppose \( T : \mathbb{R}^n \to \mathbb{R}^m \) is a one-to-one matrix transformation. Which one of the following must be true? (circle one)
      \( m = n \) \( m < n \) \( m \leq n \) \( m > n \) \( m \geq n \)

9. Which of the following transformations are onto? Circle all that apply.
   a) \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) that rotates counterclockwise by \( \frac{\pi}{12} \) radians.
   b) The transformation \( T(x) = Ax \), where \( A \) is a \( 4 \times 3 \) matrix with three pivots.
   c) \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) that reflects across the \( yz \)-plane.
Worksheet 6 (3.3-3.4)

1. If $A$ is a $3 \times 5$ matrix and $B$ is a $3 \times 2$ matrix, which of the following are defined?
   a) $A - B$
   b) $AB$
   c) $A^T B$
   d) $B^T A$
   e) $A^2$

2. $A$ is $m \times n$ matrix, $B$ is $n \times m$ matrix. Select proper answers from the box. Multiple answers are possible
   a) Take any vector $x$ in $\mathbb{R}^n$, then $ABx$ must be in:
      \[ \text{Col}(A), \text{Nul}(A), \text{Col}(B), \text{Nul}(B) \]
   b) Take any vector $x$ in $\mathbb{R}^n$, then $BAx$ must be in:
      \[ \text{Col}(A), \text{Nul}(A), \text{Col}(B), \text{Nul}(B) \]
   c) If $m > n$, then columns of $AB$ could be linearly independent, dependent
   d) If $m > n$, then columns of $BA$ could be linearly independent, dependent
   e) If $m > n$ and $Ax = 0$ has nontrivial solutions, then columns of $BA$ could be linearly independent, dependent
Supplemental (3.3-3.4)

1. Circle T if the statement is always true, and circle F otherwise.
   
a) T  F  If \( T : \mathbb{R}^n \to \mathbb{R}^n \) is linear and \( T(e_1) = T(e_2) \), then the homogeneous equation \( T(x) = 0 \) has infinitely many solutions.

b) T  F  If \( T : \mathbb{R}^n \to \mathbb{R}^n \) is a one-to-one linear transformation and \( m \neq n \), then \( T \) must not be onto.

2. Consider \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) given by
   \[
   T(x, y, z) = (x, x + z, 3x - 4y + z, x).
   
   Is \( T \) one-to-one? Justify your answer.

3. In each case, determine whether \( T \) is linear. Briefly justify.
   
a) \( T(x_1, x_2) = (x_1 - x_2, x_1 + x_2, 1) \).
   
b) \( T(x, y) = (y, x^{1/3}) \).
   
c) \( T(x, y, z) = 2x - 5z \).

4. True or false (justify your answer). Answer true if the statement is always true. Otherwise, answer false.
   
a) If \( A \) and \( B \) are matrices and the products \( AB \) and \( BA \) are both defined, then \( A \) and \( B \) must be square matrices with the same number of rows and columns.
   
b) If \( A, B, \) and \( C \) are nonzero \( 2 \times 2 \) matrices satisfying \( BA = CA \), then \( B = C \).
   
c) Suppose \( A \) is an \( 4 \times 3 \) matrix whose associated transformation \( T(x) = Ax \) is not one-to-one. Then there must be a \( 3 \times 3 \) matrix \( B \) which is not the zero matrix and satisfies \( AB = 0 \).
   
d) Suppose \( T : \mathbb{R}^6 \to \mathbb{R}^m \) and \( U : \mathbb{R}^m \to \mathbb{R}^p \) are one-to-one linear transformations. Then \( U \circ T \) is one-to-one. (What if \( U \) and \( T \) are not necessarily linear?)

5. In each case, use geometric intuition to either give an example of a matrix with the desired properties or explain why no such matrix exists.
   
a) A \( 3 \times 3 \) matrix \( P \), which is not the identity matrix or the zero matrix, and satisfies \( P^2 = P \).
   
b) A \( 2 \times 2 \) matrix \( A \) satisfying \( A^2 = I \).
   
c) A \( 2 \times 2 \) matrix \( A \) satisfying \( A^3 = -I \).
Worksheet 7 (3.5-3.6)

1. True or false (justify your answer). Answer true if the statement is always true. Otherwise, answer false.
   a) If $A$ and $B$ are $n \times n$ matrices and both are invertible, then the inverse of $AB$ is $A^{-1}B^{-1}$.

   b) If $A$ is an $n \times n$ matrix and the equation $Ax = b$ has at least one solution for each $b$ in $\mathbb{R}^n$, then the solution is unique for each $b$ in $\mathbb{R}^n$.

   c) If $A$ is an $n \times n$ matrix and the equation $Ax = b$ has at most one solution for each $b$ in $\mathbb{R}^n$, then the solution must be unique for each $b$ in $\mathbb{R}^n$.

   d) If $A$ and $B$ are invertible $n \times n$ matrices, then $A+B$ is invertible and $(A+B)^{-1} = A^{-1} + B^{-1}$.

   e) If $A$ and $B$ are $n \times n$ matrices and $ABx = 0$ has a unique solution, then $Ax = 0$ has a unique solution.

   f) If $A$ is a $3 \times 4$ matrix and $B$ is a $4 \times 2$ matrix, then the linear transformation $Z$ defined by $Z(x) = ABx$ has domain $\mathbb{R}^3$ and codomain $\mathbb{R}^2$.

   g) Suppose $A$ is an $n \times n$ matrix and every vector in $\mathbb{R}^n$ can be written as a linear combination of the columns of $A$. Then $A$ must be invertible.