Math 1553 Conceptual question list §§2.6-3.6

Solutions

Worksheet 5 (2.6-3.2)

1. Circle **TRUE** if the statement is always true, and circle **FALSE** otherwise.

a) If A is a 3×10 matrix with 2 pivots in its RREF, then $\dim(\mathrm{Nul} A) = 8$ and $\mathrm{rank}(A) = 2$.

TRUE **FALSE**

b) If A is an $m \times n$ matrix and $Ax = 0$ has only the trivial solution, then the transformation $T(x) = Ax$ is onto.

TRUE **FALSE**

c) If $\{a, b, c\}$ is a basis of a linear space V, then $\{a, a+b, b+c\}$ is a basis of V as well.

TRUE **FALSE**

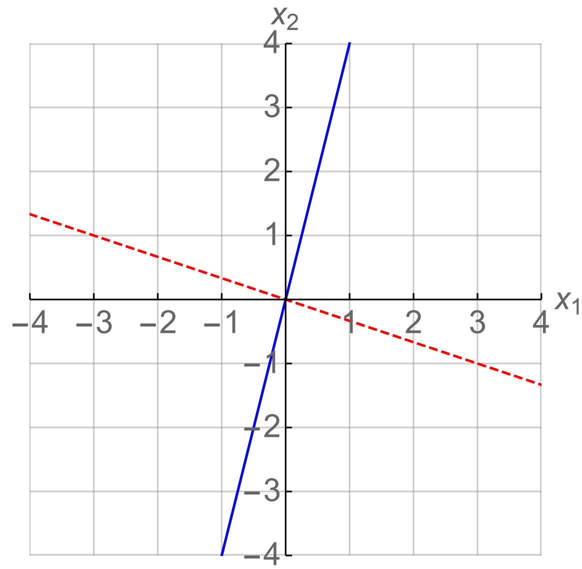
Solution.

a) True. $\mathrm{rank}(A)$ is the same as number of pivots in A. $\dim(\mathrm{Nul} A)$ is the same as the number of free variables. Moreover by the Rank Theorem, $\mathrm{rank}(A) + \dim(\mathrm{Nul} A) = 10$, so $\dim(\mathrm{Nul} A) = 10 - 2 = 8$.

b) False. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ has only the trivial solution for $Ax = 0$, but its column space is a 2-dimensional subspace of $\mathbf{R}^3$.

c) True. Because a and b are independent, $a + b$ and a are linearly independent, and furthermore a and b are in $\mathrm{Span}\{a, a+b\}$. Next, c is independent from $\{a, b\}$, so $b + c$ is independent from $\{a, a+b\}$, meaning that $\{a, a+b, b+c\}$ is independent by the increasing span criterion. Since $a, a+b, b+c$ are all clearly in $\mathrm{Span}\{a, b, c\}$, by the basis theorem $\{a, a+b, b+c\}$ also form a span for $\mathrm{Span}\{a, b, c\} = V$. Alternatively, we could notice that $a, b, c \in \mathrm{Span}\{a, a+b, b+c\}$, and since $V = \mathrm{Span}\{a, b, c\}$ it is a three-dimensional space spanned by the set of three elements $\{a, a+b, b+c\}$, those three elements must form a basis, by the basis theorem.

2. Write a matrix A so that $\mathrm{Col}(A)$ is the solid blue line and $\mathrm{Nul}(A)$ is the dotted red line drawn below.



Solution.

We'd like to design an A with the prescribed column space $\text{Span}\left\{\begin{pmatrix}1\\4\end{pmatrix}\right\}$ and null space $\text{Span}\left\{\begin{pmatrix}3\\-1\end{pmatrix}\right\}$.

We start with analyzing the null space. We can write parametric form of the null space:

$$\begin{pmatrix}x_1\\x_2\end{pmatrix} = t\begin{pmatrix}3\\-1\end{pmatrix} \quad \text{is the same as } \begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}-3x_2\\x_2\end{pmatrix}$$

Then this implies the RREF of A must be $\begin{pmatrix}1 & 3\\0 & 0\end{pmatrix}$.

Now we need to combine the information that column space is $\text{Span}\left\{\begin{pmatrix}1\\4\end{pmatrix}\right\}$. That means the second row must be 4 multiple of the first row. Therefore the second row must be $\begin{pmatrix}4 & 12\end{pmatrix}$. We conclude,

$$A = \begin{pmatrix}1 & 3\\4 & 12\end{pmatrix}$$

Note any nonzero scalar multiple of the above matrix is also a solution.

supplemental (2.6-3.2)

1. Circle **TRUE** if the statement is always true, and circle **FALSE** otherwise.

a) If A is a 3×100 matrix of rank 2, then $\dim(\text{Nul}A) = 97$.

TRUE **FALSE**

b) If A is an $m \times n$ matrix and $Ax = 0$ has only the trivial solution, then the columns of A form a basis for $\mathbf{R}^m$.

TRUE **FALSE**

c) The set $V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; x - 4z = 0 \right\}$ is a subspace of $\mathbf{R}^4$.

TRUE **FALSE**

Solution.

a) False. By the Rank Theorem, $\text{rank}(A) + \dim(\text{Nul}A) = 100$, so $\dim(\text{Nul}A) = 98$.

b) False. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ has only the trivial solution for $Ax = 0$, but its column space is a 2-dimensional subspace of $\mathbf{R}^3$.

c) True. V is $\text{Nul}(A)$ for the 1×4 matrix A below, and therefore is automatically a subspace of $\mathbf{R}^4$:

$$A = \begin{pmatrix} 1 & 0 & -4 & 0 \end{pmatrix}.$$

Alternatively, we could verify the subspace properties directly if we wished, but this is much more work!

(1) The zero vector is in V, since $0 - 4(0)0 = 0$.

(2) Let $u = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix}$ and $v = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix}$ be in V, so $x_1 - 4z_1 = 0$ and $x_2 - 4z_2 = 0$. We compute

$$u + v = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \\ w_1 + w_2 \end{pmatrix}.$$

Is $(x_1 + x_2) - 4(z_1 + z_2) = 0$? Yes, since

$$(x_1 + x_2) - 4(z_1 + z_2) = (x_1 - 4z_1) \;+\; (x_2 - 4z_2) = 0 + 0 = 0.$$

(3) If $u = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ is in V then so is cu for any scalar c:

$$cu = \begin{pmatrix} cx \\ cy \\ cz \\ cw \end{pmatrix} \qquad \text{and} \qquad cx - 4cz = c(x - 4z) = c(0) = 0.$$

2. Circle **T** if the statement is always true, and circle **F** otherwise. You do not need to explain your answer.

a) If $\{v_1, v_2, v_3, v_4\}$ is a basis for a subspace V of $\mathbf{R}^n$, then $\{v_1, v_2, v_3\}$ is a linearly independent set.

b) The solution set of a consistent matrix equation $Ax = b$ is a subspace.

c) A translate of a span is a subspace.

Solution.

a) True. If $\{v_1, v_2, v_3\}$ is linearly dependent then $\{v_1, v_2, v_3, v_4\}$ is automatically linearly dependent, which is impossible since $\{v_1, v_2, v_3, v_4\}$ is a basis for a subspace.

b) False. this is true if and only if $b = 0$, i.e., the equation is *homogeneous*, in which case the solution set is the null space of A.

c) False. A subspace must contain 0.

3. True or false (justify your answer). Answer true if the statement is *always* true. Otherwise, answer false.

a) There exists a 3×5 matrix with rank 4.

b) If A is an 9×4 matrix with a pivot in each column, then

$$\mathrm{Nul} A = \{0\}.$$

c) There exists a 4×7 matrix A such that nullity $A = 5$.

d) If $\{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbf{R}^4$, then $n = 4$.

Solution.

a) False. The rank is the dimension of the column space, which is a subspace of $\mathbf{R}^3$, hence has dimension at most 3.

b) True.

c) True. For instance,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

d) True. Any basis of $\mathbf{R}^4$ has 4 vectors.

4. **a)** True or false: If A is an $m \times n$ matrix and $\text{Nul}(A) = \mathbf{R}^n$, then $\text{Col}(A) = \{0\}$.

b) Give an example of 2×2 matrix whose column space is the same as its null space.

c) True or false: For some m, we can find an $m \times 10$ matrix A whose column span has dimension 4 and whose solution set for $Ax = 0$ has dimension 5.

Solution.

a) If $\text{Nul}(A) = \mathbf{R}^n$ then $Ax = 0$ for all x in $\mathbf{R}^n$, so the only element in $\text{Col}(A)$ is $\{0\}$. Alternatively, the rank theorem says

$$\dim(\text{Col } A) + \dim(\text{Nul } A) = n \implies \dim(\text{Col } A) + n = n \implies \dim(\text{Col } A) = 0 \implies \text{Col } A = \{0\}.$$

b) Take $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Its null space and column space are $\text{Span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$.

c) False. The rank theorem says that the dimensions of the column space ($\text{Col}A$) and homogeneous solution space ($\text{Nul}A$) add to 10, no matter what m is.

5. Fill in the blanks: If A is a 7×6 matrix and the solution set for $Ax = 0$ is a plane, then the column space of A is a ___4___ -dimensional subspace of $\mathbf{R}^{\boxed{7}}$.
Reason: $\text{rank}(A) + \text{nullity}(A) = 6 \qquad \text{rank}(A) + 2 = 6 \qquad \text{rank}(A) = 4$

6. True or false. If the statement is *always* true, answer TRUE. Otherwise, circle FALSE.

a) The matrix transformation $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ performs reflection across the x-axis in $\mathbf{R}^2$. TRUE (T reflects across the y-axis then projects onto the x-axis)

b) The matrix transformation $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ performs rotation counter-clockwise by 90° in $\mathbf{R}^2$. TRUE (T rotates clockwise 90°)

7. Let A be a 3×4 matrix with column vectors v_1, v_2, v_3, v_4, and suppose $v_2 = 2v_1 - 3v_4$. Consider the matrix transformation $T(x) = Ax$.

a) Is it possible that T is one-to-one? If yes, justify why. If no, find distinct vectors v and w so that $T(v) = T(w)$.

b) Is it possible that T is onto? Justify your answer.

Solution.

a) From the linear dependence condition we were given, we get

$$-2v_1 + v_2 + 3v_4 = 0.$$

The corresponding vector equation is just

$$\begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{so} \quad A\begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, $v = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix}$ and $w = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ both satisfy $Av = Aw = 0$, so T cannot be one-to-one.

b) Yes. If $\{v_1, v_3, v_4\}$ is linearly independent then A will have a pivot in every row and T will be onto. Such a matrix A is

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{pmatrix}.$$

8. Answer each question.

a) Suppose $S : \mathbf{R}^3 \to \mathbf{R}^2$ is the matrix transformation $S(x) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix} x$.

Is S one-to-one? NO

Is S onto? YES

b) Suppose $T : \mathbf{R}^2 \to \mathbf{R}^2$ is given by $T(x, y) = (x - y,\ x - y)$.
Is T one-to-one? NO

Is T onto? NO

c) Suppose $T : \mathbf{R}^n \to \mathbf{R}^m$ is a one-to-one matrix transformation. Which one of the following *must* be true? (cicle one)

$$m \geq n$$

9. Which of the following transformations are onto?
Circle all that apply.

a) $T : \mathbf{R}^2 \to \mathbf{R}^2$ that rotates counterclockwise by $\dfrac{\pi}{12}$ radians.

b) The transformation $T(x) = Ax$, where A is a 4×3 matrix with three pivots.

c) $T : \mathbf{R}^3 \to \mathbf{R}^3$ that reflects across the yz-plane.

Solution.

The transformations (a) and (c) are onto. Note that (b) is not onto since A doesn't have a pivot in every row. In (b), range(T) is a 3-dimensional subspace of $\mathbf{R}^4$.

Worksheet 6 (3.3-3.4)

1. If A is a 3×5 matrix and B is a 3×2 matrix, which of the following are defined?

a) $A - B$

b) AB

c) $A^T B$

d) $B^T A$

e) A^2

Solution.

Only (c) and (d).

a) $A - B$ is nonsense. In order for $A - B$ to be defined, A and B need to have the same number or rows and same number of columns.

b) AB is undefined since the number of columns of A does not equal the number of rows of B.

c) A^T is 5×3 and B is 3×2, so $A^T B$ is a 5×2 matrix.

d) B^T is 2×3 and A is 3×5, so $B^T A$ is a 2×5 matrix.

e) A^2 is nonsense (can't multiply 3×5 with another 3×5).

2. A is $m \times n$ matrix, B is $n \times m$ matrix. Select proper answers from the box. Multiple answers are possible

a) Take any vector x in $\mathbf{R}^m$, then ABx *must be* in:
$\boxed{\text{Col}(A), \quad \text{Nul}(A), \quad \text{Col}(B), \quad \text{Nul}(B)}$

b) Take any vector x in $\mathbf{R}^n$, then BAx *must be* in:
$\boxed{\text{Col}(A), \quad \text{Nul}(A), \quad \text{Col}(B), \quad \text{Nul}(B)}$

c) If $m > n$, then columns of AB could be linearly $\boxed{independent, \quad dependent}$

d) If $m > n$, then columns of BA could be linearly $\boxed{independent, \quad dependent}$

e) If $m > n$ and $Ax = 0$ has nontrivial solutions, then columns of BA could be linearly $\boxed{independent, \quad dependent}$

Solution.

Recall, AB can be computed as A multiplying every column of B. That is $AB = \begin{pmatrix} Ab_1 & Ab_2 & \cdots Ab_m \end{pmatrix}$ where $B = \begin{pmatrix} b_1 & b_2 & \cdots b_m \end{pmatrix}$.

a) $\boxed{\text{Col}(A)}$. Denote $w := Bx$, which is a vector in $\mathbf{R}^n$. $ABx = A(Bx)$ is multiplying A with w which will end up with "linear combination of columns of A". So ABx is in Col(A).

b) $\boxed{\text{Col}(B)}$. Similarly, $BAx = B(Ax)$ is multiplying B with Ax, a vector in R^m, which will end up with "linear combination of columns of B". So BAx is in Col(B).

c) $\boxed{dependent}$. Since $m > n$ means A matrix can have at most n pivots. So $dim(\text{Col}(A)) \leq n$. Notice from first question we know $\text{Col}(AB) \subset \text{Col}(A)$ which has dimension at most n. That means AB can have at most n pivots. But AB is $m \times m$ matrix, then columns of AB must be dependent.

d) $\boxed{independent, dependent}$. Both are possible. Since $m > n$ means B matrix can have at most n pivots. then $\text{Col}(BA) \subset \text{Col}(B)$ means BA can have at most n pivots. Since BA is $n \times n$ matrix, then the columns of BA will be linearly independent when there are n pivots or linearly dependent when there are less than n pivots. Here are two examples.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \text{then } BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \text{then } BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

e) $\boxed{dependent}$. From the second example above, BA has dependent columns, we know "dependent" is one possible answer. Now to see if "independent" is also possible, we need to find out if BA could have n pivots.

Since $Ax = 0$ has nontrivial solution say x^*, then x^* is also a nontrivial solution of $BAx = 0$. That means BA has free variables, and it can not have n pivots. So columns of BA must be linearly dependent.

To summarize what we are actually study here, there are several relations between these subspaces.

$$\text{Col}(AB) \subset \text{Col}(A);$$
$$\text{Col}(BA) \subset \text{Col}(B);$$
$$\text{Nul}(A) \subset \text{Nul}(BA);$$
$$\text{Nul}(B) \subset \text{Nul}(AB);$$

Supplemental (3.3-3.4)

1. Circle **T** if the statement is always true, and circle **F** otherwise.

a) **T** **F** If $T : \mathbf{R}^n \to \mathbf{R}^n$ is linear and $T(e_1) = T(e_2)$, then the homogeneous equation $T(x) = 0$ has infinitely many solutions.

b) **T** **F** If $T : \mathbf{R}^n \to \mathbf{R}^m$ is a one-to-one linear transformation and $m \neq n$, then T must not be onto.

Solution.

a) True. The matrix transformation $T(x) = Ax$ is not one-to-one, so $Ax = 0$ has infinitely many solutions. For example, $e_1 - e_2$ is a non-trivial solution to $Ax = 0$ since $A(e_1 - e_2) = Ae_1 - Ae_2 = 0$.

b) True. Let A be the $m \times n$ standard matrix for T. If T is both one-to-one and onto then T must have a pivot in each column and in each row, which is only possible when A is a square matrix ($m = n$).

2. Consider $T : \mathbf{R}^3 \to \mathbf{R}^3$ given by

$$T(x, y, z) = (x,\ x+z,\ 3x - 4y + z,\ x).$$

Is T one-to-one? Justify your answer.

Solution.

One approach: We form the standard matrix A for T:

$$A = \begin{pmatrix} T(e_1) & T(e_2) & T(e_3) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 3 & -4 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We row-reduce A until we determine its pivot columns

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 3 & -4 & 1 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow[R_3 = R_3 - 3R_1,\ R_4 = R_4 - R_1]{R_2 = R_2 - R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

A has a pivot in every column, so T is one-to-one.

Alternative approach: T is a linear transformation, so it is one-to-one if and only if the equation $T(x, y, z) = (0, 0, 0)$ has only the trivial solution.
If $T(x, y, z) = (x,\ x+z,\ 3x - 4y + z,\ x) = (0, 0, 0, 0)$ then $x = 0$, and

$$x + z = 0 \implies 0 + z = 0 \implies z = 0, \text{ and finally}$$

$$3x - 4y + z = 0 \implies 0 - 4y + 0 = 0 \implies y = 0,$$

so the trivial solution $x = y = z = 0$ is the only solution the homogeneous equation. Therefore, T is one-to-one.

3. In each case, determine whether T is linear. Briefly justify.

a) $T(x_1, x_2) = (x_1 - x_2, x_1 + x_2, 1)$.

b) $T(x, y) = (y, x^{1/3})$.

c) $T(x, y, z) = 2x - 5z$.

Solution.

a) Not linear. $T(0,0) = (0,0,1) \neq (0,0,0)$.

b) Not linear. The $x^{1/3}$ term gives it away. $T(0,2) = (0, 2^{1/3})$ but $2T(0,1) = (0,2)$.

c) Linear. In fact, $T(v) = Av$ where

$$A = \begin{pmatrix} 2 & 0 & -5 \end{pmatrix}.$$

4. True or false (justify your answer). Answer true if the statement is *always* true. Otherwise, answer false.

a) If A and B are matrices and the products AB and BA are both defined, then A and B must be square matrices with the same number of rows and columns.

b) If A, B, and C are nonzero 2×2 matrices satisfying $BA = CA$, then $B = C$.

c) Suppose A is an 4×3 matrix whose associated transformation $T(x) = Ax$ is not one-to-one. Then there must be a 3×3 matrix B which is not the zero matrix and satisfies $AB = 0$.

d) Suppose $T : \mathbf{R}^n \to \mathbf{R}^m$ and $U : \mathbf{R}^m \to \mathbf{R}^p$ are one-to-one linear transformations. Then $U \circ T$ is one-to-one. (What if U and T are not necessarily linear?)

Solution.

a) False. For example, if A is any 2×3 matrix and B is any 3×2 matrix, then AB and BA are both defined.

b) False. Take $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $BA = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $BC = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, but $B \neq C$.

c) True. If T is not one-to-one then there is a non-zero vector v in $\mathbf{R}^3$ so that

$$Av = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The 3×3 matrix $B = \begin{pmatrix} | & | & | \\ v & v & v \\ | & | & | \end{pmatrix}$ satisfies

$$AB = \begin{pmatrix} | & | & | \\ Av & Av & Av \\ | & | & | \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

d) True. Recall that a transformation S is one-to-one if $S(x) = S(y)$ implies $x = y$ (the same outputs implies the same inputs). Suppose that $U \circ T(x) = U \circ T(y)$. Then $U(T(x)) = U(T(y))$, so since U is one-to-one, we have $T(x) = T(y)$. Since T is one-to-one, this implies $x = y$. Therefore, $U \circ T$ is one-to-one. Note that this argument does not use the assumption that U and T are linear transformations.

Alternative: We'll show that $U \circ T(x) = 0$ has only the trivial solution. Let A be the matrix for U and B be the matrix for T, and suppose x is a vector satisfying $(U \circ T)(x) = 0$. In terms of matrix multiplication, this is equivalent to $ABx = 0$. Since U is one-to-one, the only solution to $Av = 0$ is $v = 0$, so $A(Bx) = 0 \implies Bx = 0$.

Since T is one-to-one, we know that $Bx = 0 \implies x = 0$. Therefore, the equation $(U \circ T)(x) = 0$ has only the trivial solution.

5. In each case, use geometric intuition to either give an example of a matrix with the desired properties or explain why no such matrix exists.

a) A 3×3 matrix P, which is not the identity matrix or the zero matrix, and satisfies $P^2 = P$.

b) A 2×2 matrix A satisfying $A^2 = I$.

c) A 2×2 matrix A satisfying $A^3 = -I$.

Solution.

a) Take P to be the natural projection onto the xy-plane in $\mathbf{R}^3$, so $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

If you apply P to a vector then the result will be within the xy-plane of $\mathbf{R}^3$, so applying P a second time won't change anything, hence $P^2 = P$.

b) Take A to be matrix for reflection across the line $y = x$, so $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since A swaps the x and y coordinates, repeating A will swap them back to their original positions, so $AA = I$.

c) Note that $-I$ is the matrix that rotates counterclockwise by 180°, so we need a transformation that will give you counterclockwise rotation by 180° if you do it three times. One such matrix is the rotation matrix for 60° counterclockwise,

$$A = \begin{pmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}.$$

Another such matrix is $A = -I$.

Worksheet 7 (3.5-3.6)

1. True or false (justify your answer). Answer true if the statement is *always* true. Otherwise, answer false.

a) If A and B are $n \times n$ matrices and both are invertible, then the inverse of AB is $A^{-1}B^{-1}$.

b) If A is an $n \times n$ matrix and the equation $Ax = b$ has at least one solution for each b in $\mathbf{R}^n$, then the solution is *unique* for each b in $\mathbf{R}^n$.

c) If A is an $n \times n$ matrix and the equation $Ax = b$ has at most one solution for each b in $\mathbf{R}^n$, then the solution must be *unique* for each b in $\mathbf{R}^n$.

d) If A and B are invertible $n \times n$ matrices, then $A+B$ is invertible and $(A+B)^{-1} = A^{-1} + B^{-1}$.

e) If A and B are $n \times n$ matrices and $ABx = 0$ has a unique solution, then $Ax = 0$ has a unique solution.

f) If A is a 3×4 matrix and B is a 4×2 matrix, then the linear transformation Z defined by $Z(x) = ABx$ has domain $\mathbf{R}^3$ and codomain $\mathbf{R}^2$.

g) Suppose A is an $n \times n$ matrix and every vector in $\mathbf{R}^n$ can be written as a linear combination of the columns of A. Then A must be invertible.

Solution.

a) False. $(AB)^{-1} = B^{-1}A^{-1}$.

b) True. The first part says the transformation $T(x) = Ax$ is onto. Since A is $n \times n$, then it has n pivots. This is the same as saying A is invertible, and there is no free variable. Therefore, the equation $Ax = b$ has exactly one solution for each b in $\mathbf{R}^n$.

c) True. The first part says the transformation $T(x) = Ax$ is one-to-one (namely not multiple-to-one). Since A is $n \times n$, then it has n pivots. Then there is no free variable. Therefore, the equation $Ax = b$ has exactly one solution for each b in $\mathbf{R}^n$.

d) False. $A + B$ might not be invertible in the first place. For example, if $A = I_2$ and $B = -I_2$ then $A + B = 0$ which is not invertible. Even in the case when $A + B$ is invertible, it still might not be true that $(A + B)^{-1} = A^{-1} + B^{-1}$. For example, $(I_2 + I_2)^{-1} = (2I_2)^{-1} = \frac{1}{2}I_2$, whereas $(I_2)^{-1} + (I_2)^{-1} = I_2 + I_2 = 2I_2$.

e) True. According to the Invertible Matrix Theorem, the product AB is invertible. This implies A is invertible, with inverse $B(AB)^{-1}$:

$$A \cdot B(AB)^{-1} = (AB)(AB)^{-1} = I_n.$$

f) False. In order for Bx to make sense, x must be in $\mathbf{R}^2$, and so Bx is in $\mathbf{R}^4$ and $A(Bx)$ is in $\mathbf{R}^3$. Therefore, the domain of Z is $\mathbf{R}^2$ and the codomain of Z is $\mathbf{R}^3$.

g) True. If the columns of A span $\mathbf{R}^n$, then A is invertible by the Invertible Matrix Theorem. We can also see this directly without quoting the IMT:

If the columns of A span $\mathbf{R}^n$, then A has n pivots, so A has a pivot in each row and column, hence its matrix transformation $T(x) = Ax$ is one-to-one and onto and thus invertible. Therefore, A is invertible.