Supplemental problems: §§2.6, 2.7, 2.9

1. Circle **TRUE** if the statement is always true, and circle **FALSE** otherwise.

a) If A is a 3×100 matrix of rank 2, then $\dim(\mathrm{Nul}A) = 97$.

TRUE **FALSE**

b) If A is an $m \times n$ matrix and $Ax = 0$ has only the trivial solution, then the columns of A form a basis for $\mathbf{R}^m$.

TRUE **FALSE**

c) The set $V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; x - 4z = 0 \right\}$ is a subspace of $\mathbf{R}^4$.

TRUE **FALSE**

Solution.

a) False. By the Rank Theorem, $\mathrm{rank}(A) + \dim(\mathrm{Nul}A) = 100$, so $\dim(\mathrm{Nul}A) = 98$.

b) False. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ has only the trivial solution for $Ax = 0$, but its column space is a 2-dimensional subspace of $\mathbf{R}^3$.

c) True. V is $\mathrm{Nul}(A)$ for the 1×4 matrix A below, and therefore is automatically a subspace of $\mathbf{R}^4$:

$$A = \begin{pmatrix} 1 & 0 & -4 & 0 \end{pmatrix}.$$

Alternatively, we could verify the subspace properties directly if we wished, but this is much more work!

(1) The zero vector is in V, since $0 - 4(0)0 = 0$.

(2) Let $u = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix}$ and $v = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix}$ be in V, so $x_1 - 4z_1 = 0$ and $x_2 - 4z_2 = 0$. We compute

$$u + v = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \\ w_1 + w_2 \end{pmatrix}.$$

Is $(x_1 + x_2) - 4(z_1 + z_2) = 0$? Yes, since

$$(x_1 + x_2) - 4(z_1 + z_2) = (x_1 - 4z_1) \; + \; (x_2 - 4z_2) = 0 + 0 = 0.$$

(3) If $u = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ is in V then so is cu for any scalar c:

$$cu = \begin{pmatrix} cx \\ cy \\ cz \\ cw \end{pmatrix} \quad \text{and} \quad cx - 4cz = c(x-4z) = c(0) = 0.$$

2. Write a matrix A so that $\operatorname{Col} A = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}\right\}$ and $\operatorname{Nul} A$ is the xz-plane.

Solution.

Many examples are possible. We'd like to design an A with the prescribed column span, so that $\left(A \mid 0\right)$ will have free variables x_1 and x_3. One way to do this is simply to leave the x_1 and x_3 columns blank, and make the second column $\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$. This guarantees that A destroys the xz-plane and has the column span required.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

An alternative method for finding the same matrix: Write $A = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$. We want the column span to be the span of $\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$ and we want

$$A\begin{pmatrix} x \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}\begin{pmatrix} x \\ 0 \\ z \end{pmatrix} = xv_1 + zv_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{for all } x \text{ and } z.$$

One way to do this is choose $v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, and $v_2 = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$.

3. Circle **T** if the statement is always true, and circle **F** otherwise. You do not need to explain your answer.

a) If $\{v_1, v_2, v_3, v_4\}$ is a basis for a subspace V of $\mathbf{R}^n$, then $\{v_1, v_2, v_3\}$ is a linearly independent set.

b) The solution set of a consistent matrix equation $Ax = b$ is a subspace.

c) A translate of a span is a subspace.

Solution.

a) True. If $\{v_1, v_2, v_3\}$ is linearly dependent then $\{v_1, v_2, v_3, v_4\}$ is automatically linearly dependent, which is impossible since $\{v_1, v_2, v_3, v_4\}$ is a basis for a subspace.

b) False. this is true if and only if $b = 0$, i.e., the equation is *homogeneous*, in which case the solution set is the null space of A.

c) False. A subspace must contain 0.

4. True or false (justify your answer). Answer true if the statement is *always* true. Otherwise, answer false.

a) There exists a 3×5 matrix with rank 4.

b) If A is an 9×4 matrix with a pivot in each column, then

$$\mathrm{Nul} A = \{0\}.$$

c) There exists a 4×7 matrix A such that nullity $A = 5$.

d) If $\{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbf{R}^4$, then $n = 4$.

Solution.

a) False. The rank is the dimension of the column space, which is a subspace of $\mathbf{R}^3$, hence has dimension at most 3.

b) True.

c) True. For instance,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

d) True. Any basis of $\mathbf{R}^4$ has 4 vectors.

5. Find bases for the column space and the null space of

$$A = \begin{pmatrix} 0 & 1 & -3 & 1 & 0 \\ 1 & -1 & 8 & -7 & 1 \\ -1 & -2 & 1 & 4 & -1 \end{pmatrix}.$$

Solution.

The RREF of $\left(A \mid 0\right)$ is

$$\left(\begin{array}{ccccc|c} 1 & 0 & 5 & -6 & 1 & 0 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

so x_3, x_4, x_5 are free, and

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -5x_3 + 6x_4 - x_5 \\ 3x_3 - x_4 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_3 \begin{pmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 6 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, a basis for Nul A is $\left\{ \begin{pmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

To find a basis for Col A, we use the pivot columns as they were written in the *original* matrix A, not its RREF. These are the first two columns:

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right\}.$$

6. Find a basis for the subspace V of $\mathbf{R}^4$ given by

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; x + 2y - 3z + w = 0 \right\}.$$

Solution.

V is Nul A for the 1×4 matrix $A = \begin{pmatrix} 1 & 2 & -3 & 1 \end{pmatrix}$. The augmented matrix $\left(A \mid 0\right) = \left(\begin{matrix} 1 & 2 & -3 & 1 \end{matrix} \mid 0\right)$ gives $x = -2y + 3z - w$ where y, z, w are free variables. The parametric vector form for the solution set to $Ax = 0$ is

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2y + 3z - w \\ y \\ z \\ w \end{pmatrix} = y \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, a basis for V is

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

7. **a)** True or false: If A is an $m \times n$ matrix and $\text{Nul}(A) = \mathbf{R}^n$, then $\text{Col}(A) = \{0\}$.

b) Give an example of 2×2 matrix whose column space is the same as its null space.

c) True or false: For some m, we can find an $m \times 10$ matrix A whose column span has dimension 4 and whose solution set for $Ax = 0$ has dimension 5.

Solution.

a) If $\mathrm{Nul}(A) = \mathbf{R}^n$ then $Ax = 0$ for all x in $\mathbf{R}^n$, so the only element in $\mathrm{Col}(A)$ is $\{0\}$. Alternatively, the rank theorem says

$$\dim(\mathrm{Col}\ A) + \dim(\mathrm{Nul}\ A) = n \implies \dim(\mathrm{Col}\ A) + n = n \implies \dim(\mathrm{Col}\ A) = 0 \implies \mathrm{Col}\ A = \{0\}.$$

b) Take $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Its null space and column space are $\mathrm{Span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$.

c) False. The rank theorem says that the dimensions of the column space ($\mathrm{Col}A$) and homogeneous solution space ($\mathrm{Nul}A$) add to 10, no matter what m is.

8. Suppose V is a 3-dimensional subspace of $\mathbf{R}^5$ containing $\begin{pmatrix} 1 \\ -4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 9 \\ 8 \\ 1 \\ 0 \\ 1 \end{pmatrix}$.

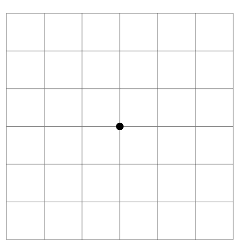
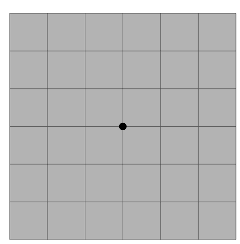
Is $\left\{ \begin{pmatrix} 1 \\ -4 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 9 \\ 8 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ a basis for V? Justify your answer.

Solution.

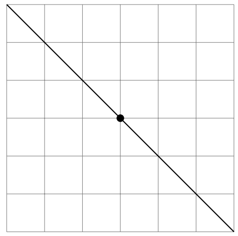
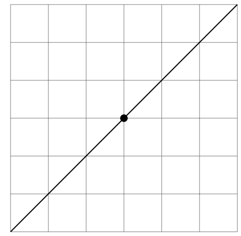
Yes. The Basis Theorem says that since we know $\dim(V) = 3$, our three vectors will form a basis for V if and only if they are linearly independent.

Call the vectors v_1, v_2, v_3. It is very little work to show that the matrix $A = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ has a pivot in every column, so the vectors are linearly independent.

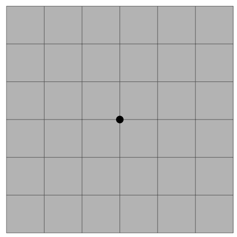
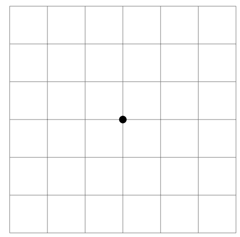
9. **a)** Write a 2×2 matrix A with **rank** 2, and draw pictures of $\mathrm{Nul}A$ and $\mathrm{Col}A$.

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Nul A =  Col A = 

b) Write a 2×2 matrix B with **rank** 1, and draw pictures of $\mathrm{Nul}B$ and $\mathrm{Col}B$.

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ Nul B =  Col B = 

c) Write a 2×2 matrix C with **rank** 0, and draw pictures of Nul C and Col C.

$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ Nul C =  Col C = 

(In the grids, the dot is the origin.)