Supplemental problems: §§2.6, 2.7, 2.9

1. Circle **TRUE** if the statement is always true, and circle **FALSE** otherwise.
   a) If $A$ is a $3 \times 100$ matrix of rank 2, then $\text{dim}(\text{Nul}A) = 97$.
   
   **TRUE**      **FALSE**

   b) If $A$ is an $m \times n$ matrix and $Ax = 0$ has only the trivial solution, then the columns of $A$ form a basis for $\mathbb{R}^m$.
   
   **TRUE**      **FALSE**

   c) The set $V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \mid x - 4z = 0 \right\}$ is a subspace of $\mathbb{R}^4$.
   
   **TRUE**      **FALSE**

Solution.

a) False. By the Rank Theorem, $\text{rank}(A) + \text{dim}(\text{Nul}A) = 100$, so $\text{dim}(\text{Nul}A) = 98$.

b) False. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ has only the trivial solution for $Ax = 0$, but its column space is a 2-dimensional subspace of $\mathbb{R}^3$.

c) True. $V$ is $\text{Nul}(A)$ for the $1 \times 4$ matrix $A$ below, and therefore is automatically a subspace of $\mathbb{R}^4$:

   $A = \begin{pmatrix} 1 & 0 & -4 & 0 \end{pmatrix}$.

   Alternatively, we could verify the subspace properties directly if we wished, but this is much more work!

   (1) The zero vector is in $V$, since $0 - 4(0)0 = 0$.

   (2) Let $u = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix}$ and $v = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix}$ be in $V$, so $x_1 - 4z_1 = 0$ and $x_2 - 4z_2 = 0$.

   We compute

   $u + v = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \\ w_1 + w_2 \end{pmatrix}$.

   Is $(x_1 + x_2) - 4(z_1 + z_2) = 0$? Yes, since

   $(x_1 + x_2) - 4(z_1 + z_2) = (x_1 - 4z_1) + (x_2 - 4z_2) = 0 + 0 = 0$. 


(3) If \( u = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \) is in \( V \) then so is \( cu \) for any scalar \( c \):
\[
cu = \begin{pmatrix} cx \\ cy \\ cz \\ cw \end{pmatrix}
\]
and \( cx - 4cz = c(x - 4z) = c(0) = 0. \)

2. Write a matrix \( A \) so that \( \text{Col}A = \text{Span}\left\{ \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \right\} \) and \( \text{Nul}A \) is the \( xz \)-plane.

Solution.
Many examples are possible. We’d like to design an \( A \) with the prescribed column span, so that \( (A \mid 0) \) will have free variables \( x_1 \) and \( x_3 \). One way to do this is simply to leave the \( x_1 \) and \( x_3 \) columns blank, and make the second column \( \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \). This guarantees that \( A \) destroys the \( xz \)-plane and has the column span required.
\[
A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\]
An alternative method for finding the same matrix: Write \( A = (v_1 \quad v_2 \quad v_3) \). We want the column span to be the span of \( \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \) and we want
\[
A \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} = (v_1 \quad v_2 \quad v_3) \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} = xv_1 + zv_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ for all } x \text{ and } z.
\]
One way to do this is choose \( v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \) and \( v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \), and \( v_2 = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \).

3. Circle T if the statement is always true, and circle F otherwise. You do not need to explain your answer.
   a) If \( \{v_1, v_2, v_3, v_4\} \) is a basis for a subspace \( V \) of \( \mathbb{R}^n \), then \( \{v_1, v_2, v_3\} \) is a linearly independent set.
   b) The solution set of a consistent matrix equation \( Ax = b \) is a subspace.
   c) A translate of a span is a subspace.

Solution.
a) True. If \( \{v_1, v_2, v_3\} \) is linearly dependent then \( \{v_1, v_2, v_3, v_4\} \) is automatically linearly dependent, which is impossible since \( \{v_1, v_2, v_3, v_4\} \) is a basis for a subspace.

b) False. This is true if and only if \( b = 0 \), i.e., the equation is homogeneous, in which case the solution set is the null space of \( A \).

c) False. A subspace must contain 0.

4. True or false (justify your answer). Answer true if the statement is always true. Otherwise, answer false.
   a) There exists a \( 3 \times 5 \) matrix with rank 4.
   b) If \( A \) is an \( 9 \times 4 \) matrix with a pivot in each column, then
      \[ \text{Nul} A = \{0\}. \]
   c) There exists a \( 4 \times 7 \) matrix \( A \) such that nullity \( A = 5 \).
   d) If \( \{v_1, v_2, \ldots, v_n\} \) is a basis for \( \mathbb{R}^4 \), then \( n = 4 \).

Solution.

a) False. The rank is the dimension of the column space, which is a subspace of \( \mathbb{R}^3 \), hence has dimension at most 3.

b) True.

c) True. For instance,
\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

d) True. Any basis of \( \mathbb{R}^4 \) has 4 vectors.

5. Find bases for the column space and the null space of
\[
A = \begin{pmatrix}
0 & 1 & -3 & 1 & 0 \\
1 & -1 & 8 & -7 & 1 \\
-1 & -2 & 1 & 4 & -1
\end{pmatrix}.
\]

Solution.

The RREF of \((A \mid 0)\) is
\[
\begin{pmatrix}
1 & 0 & 0 & -6 & 1 & 0 \\
0 & 1 & -3 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
so $x_3, x_4, x_5$ are free, and
\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5
\end{pmatrix}
= 
\begin{pmatrix}
  -5x_3 + 6x_4 - x_5 \\
  3x_3 - x_4 \\
  x_3 \\
  x_4 \\
  x_5
\end{pmatrix}
= x_3 \begin{pmatrix}
  -5 \\
  3 \\
  1 \\
  0 \\
  0
\end{pmatrix}
+ x_4 \begin{pmatrix}
  6 \\
  -1 \\
  0 \\
  1 \\
  0
\end{pmatrix}
+ x_5 \begin{pmatrix}
  -1 \\
  0 \\
  0 \\
  1 \\
  0
\end{pmatrix}.
\]
Therefore, a basis for Nul $A$ is
\[
\left\{ \begin{pmatrix}
  -5 \\
  3 \\
  1 \\
  0 \\
  0
\end{pmatrix}, 
\begin{pmatrix}
  6 \\
  -1 \\
  0 \\
  1 \\
  0
\end{pmatrix}, 
\begin{pmatrix}
  -1 \\
  0 \\
  0 \\
  1 \\
  0
\end{pmatrix} \right\}.
\]
To find a basis for Col $A$, we use the pivot columns as they were written in the original matrix $A$, not its RREF. These are the first two columns:
\[
\left\{ \begin{pmatrix}
  0 \\
  1 \\
  -1 \\
  -2
\end{pmatrix},
\begin{pmatrix}
  1 \\
  -1 \\
  0 \\
  1
\end{pmatrix} \right\}.
\]

6. Find a basis for the subspace $V$ of $\mathbb{R}^4$ given by
\[
V = \left\{ \begin{pmatrix}
  x \\
  y \\
  z \\
  w
\end{pmatrix} \in \mathbb{R}^4 \mid x + 2y - 3z + w = 0 \right\}.
\]

Solution.

$V$ is Nul $A$ for the $1 \times 4$ matrix $A = \begin{pmatrix} 1 & 2 & -3 & 1 \end{pmatrix}$. The augmented matrix $\begin{pmatrix} A \mid 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -3 & 1 & 0 \end{pmatrix}$ gives $x = -2y + 3z - w$ where $y, z, w$ are free variables. The parametric vector form for the solution set to $Ax = 0$ is
\[
\begin{pmatrix}
  x \\
  y \\
  z \\
  w
\end{pmatrix}
= \begin{pmatrix}
  -2y + 3z - w \\
  y \\
  z \\
  w
\end{pmatrix}
= y \begin{pmatrix}
  -2 \\
  1 \\
  0 \\
  0
\end{pmatrix}
+ z \begin{pmatrix}
  3 \\
  0 \\
  1 \\
  0
\end{pmatrix}
+ w \begin{pmatrix}
  -1 \\
  0 \\
  0 \\
  1
\end{pmatrix}.
\]
Therefore, a basis for $V$ is
\[
\left\{ \begin{pmatrix}
  -2 \\
  1 \\
  0 \\
  0
\end{pmatrix},
\begin{pmatrix}
  3 \\
  0 \\
  1 \\
  0
\end{pmatrix},
\begin{pmatrix}
  -1 \\
  0 \\
  0 \\
  1
\end{pmatrix} \right\}.
\]

7. a) True or false: If $A$ is an $m \times n$ matrix and Nul$(A) = \mathbb{R}^n$, then Col$(A) = \{0\}$.

b) Give an example of a $2 \times 2$ matrix whose column space is the same as its null space.
c) True or false: For some $m$, we can find an $m \times 10$ matrix $A$ whose column span has dimension 4 and whose solution set for $Ax = 0$ has dimension 5.

Solution.

a) If $\text{Nul}(A) = \mathbb{R}^n$ then $Ax = 0$ for all $x$ in $\mathbb{R}^n$, so the only element in $\text{Col}(A)$ is $\{0\}$. Alternatively, the rank theorem says

$$\dim(\text{Col}(A)) + \dim(\text{Nul}(A)) = n \implies \dim(\text{Col}(A)) + n = n \implies \dim(\text{Col}(A)) = 0 \implies \text{Col}(A) = \{0\}.$$ 

b) Take $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Its null space and column space are $\text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$.

c) False. The rank theorem says that the dimensions of the column space ($\text{Col}(A)$) and homogeneous solution space ($\text{Nul}(A)$) add to 10, no matter what $m$ is.

8. Suppose $V$ is a 3-dimensional subspace of $\mathbb{R}^5$ containing \( \begin{pmatrix} 1 \\ -4 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 9 \\ 8 \\ 1 \\ 0 \\ 1 \end{pmatrix} \). Is $\left\{ \begin{pmatrix} 1 \\ -4 \\ 0 \\ 0 \\ 0 \\ \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 9 \\ 8 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ a basis for $V$? Justify your answer.

Solution.

Yes. The Basis Theorem says that since we know $\dim(V) = 3$, our three vectors will form a basis for $V$ if and only if they are linearly independent.

Call the vectors $v_1, v_2, v_3$. It is very little work to show that the matrix $A = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ has a pivot in every column, so the vectors are linearly independent.

9. a) Write a $2 \times 2$ matrix $A$ with rank 2, and draw pictures of $\text{Nul}(A)$ and $\text{Col}(A)$.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{Nul}(A) = \quad \text{Col}(A) =$$

b) Write a $2 \times 2$ matrix $B$ with rank 1, and draw pictures of $\text{Nul}(B)$ and $\text{Col}(B)$. 

\( A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{Nul} \ B = \quad \text{Col} \ B = \)

\[ \begin{array}{c}
\text{Nul} \ C = \\
\text{Col} \ C = \\
\text{(In the grids, the dot is the origin.)}
\end{array} \]

\( A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{Nul} \ C = \quad \text{Col} \ C = \)

\( \text{c) Write a } 2 \times 2 \text{ matrix } C \text{ with rank 0, and draw pictures of Nul} \ C \text{ and Col} \ C. \)