Chapter 2

Systems of Linear Equations: Geometry
We want to think about the *algebra* in linear algebra (systems of equations and their solution sets) in terms of *geometry* (points, lines, planes, etc).

\[
\begin{align*}
x - 3y & = -3 \\
2x + y & = 8
\end{align*}
\]

This will give us better insight into the properties of systems of equations and their solution sets.

**Remember:** I expect you to be able to draw pictures!
Section 2.1

Vectors
We have been drawing elements of $\mathbb{R}^n$ as points in the line, plane, space, etc. We can also draw them as arrows.

**Definition**

A **point** is an element of $\mathbb{R}^n$, drawn as a point (a dot).

A **vector** is an element of $\mathbb{R}^n$, drawn as an arrow. When we think of an element of $\mathbb{R}^n$ as a vector, we’ll usually write it vertically, like a matrix with one column:

$$v = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

The difference is purely psychological: *points and vectors are just lists of numbers.*
So why make the distinction?

A vector need not start at the origin: *it can be located anywhere!* In other words, an arrow is determined by its length and its direction, not by its location.

These arrows all represent the vector \((1, 2)\).

However, unless otherwise specified, we’ll assume a vector starts at the origin.
Vector Algebra

Definition

- We can add two vectors together:

\[
\begin{pmatrix}
  a \\
  b \\
  c
\end{pmatrix}
+ \begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix}
= \begin{pmatrix}
  a + x \\
  b + y \\
  c + z
\end{pmatrix}.
\]

- We can multiply, or scale, a vector by a real number \(c\):

\[
c \begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix}
= \begin{pmatrix}
  c \cdot x \\
  c \cdot y \\
  c \cdot z
\end{pmatrix}.
\]

We call \(c\) a scalar to distinguish it from a vector. If \(v\) is a vector and \(c\) is a scalar, \(cv\) is called a scalar multiple of \(v\).

(And likewise for vectors of length \(n\).) For instance,

\[
\begin{pmatrix}
  1 \\
  2 \\
  3
\end{pmatrix}
+ \begin{pmatrix}
  4 \\
  5 \\
  6
\end{pmatrix}
= \begin{pmatrix}
  5 \\
  7 \\
  9
\end{pmatrix}
\quad\text{and}\quad
-2 \begin{pmatrix}
  1 \\
  2 \\
  3
\end{pmatrix}
= \begin{pmatrix}
  -2 \\
  -4 \\
  -6
\end{pmatrix}.
\]
The parallelogram law for vector addition

Geometrically, the sum of two vectors $v, w$ is obtained as follows: place the tail of $w$ at the head of $v$. Then $v + w$ is the vector whose tail is the tail of $v$ and whose head is the head of $w$. Doing this both ways creates a parallelogram. For example,

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}.$$

Why? The width of $v + w$ is the sum of the widths, and likewise with the heights. [interactive]

Vector subtraction

Geometrically, the difference of two vectors $v, w$ is obtained as follows: place the tail of $v$ and $w$ at the same point. Then $v - w$ is the vector from the head of $w$ to the head of $v$. For example,

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$$

Why? If you add $v - w$ to $w$, you get $v$. [interactive]

This works in higher dimensions too!
Scalar Multiplication: Geometry

Scalar multiples of a vector
These have the same direction but a different length.

Some multiples of $v$.
$-\frac{1}{2}v$
$0v$
$v$
$2v$

All multiples of $v$.
$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
$2v = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$
$-\frac{1}{2}v = \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}$
$0v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

[interactive]

So the scalar multiples of $v$ form a line.
We can add and scalar multiply in the same equation:

\[ w = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p \]

where \( c_1, c_2, \ldots, c_p \) are scalars, \( v_1, v_2, \ldots, v_p \) are vectors in \( \mathbb{R}^n \), and \( w \) is a vector in \( \mathbb{R}^n \).

**Definition**

We call \( w \) a **linear combination** of the vectors \( v_1, v_2, \ldots, v_p \). The scalars \( c_1, c_2, \ldots, c_p \) are called the **weights** or **coefficients**.

**Example**

Let \( v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \) and \( w = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \).

What are some linear combinations of \( v \) and \( w \)?

- \( v + w \)
- \( v - w \)
- \( 2v + 0w \)
- \( 2w \)
- \( -v \)

[interactive: 2 vectors] [interactive: 3 vectors]
Is there any vector in $\mathbb{R}^2$ that is not a linear combination of $v$ and $w$?

No: in fact, every vector in $\mathbb{R}^2$ is a combination of $v$ and $w$.

(The purple lines are to help measure how much of $v$ and $w$ you need to get to a given point.)
What are some linear combinations of \( \mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \)?

- \( \frac{3}{2} \mathbf{v} \)
- \( -\frac{1}{2} \mathbf{v} \)
- \( \ldots \)

What are all linear combinations of \( \mathbf{v} \)?
All vectors \( c\mathbf{v} \) for \( c \) a real number. I.e., all scalar multiples of \( \mathbf{v} \). These form a line.

**Question**
What are all linear combinations of

\[ \mathbf{v} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \]?

**Answer:** The line which contains both vectors.

What’s different about this example and the one on the poll?

[interactive]
Section 2.2

Vector Equations and Spans
Solve the following system of linear equations:

\[
\begin{align*}
    x - y &= 8 \\
    2x - 2y &= 16 \\
    6x - y &= 3.
\end{align*}
\]

We can write all three equations at once as vectors:

\[
\begin{pmatrix}
    x - y \\
    2x - 2y \\
    6x - y
\end{pmatrix}
= \begin{pmatrix}
    8 \\
    16 \\
    3
\end{pmatrix}.
\]

We can write this as a linear combination:

\[
x \begin{pmatrix}
    1 \\
    2 \\
    6
\end{pmatrix}
+ y \begin{pmatrix}
    -1 \\
    -2 \\
    -1
\end{pmatrix}
= \begin{pmatrix}
    8 \\
    16 \\
    3
\end{pmatrix}.
\]

So we are asking:

**Question:** Is \( \begin{pmatrix}
    8 \\
    16 \\
    3
\end{pmatrix} \) a linear combination of \( \begin{pmatrix}
    1 \\
    2 \\
    6
\end{pmatrix} \) and \( \begin{pmatrix}
    -1 \\
    -2 \\
    -1
\end{pmatrix} \)?
Systems of Linear Equations

Continued

\[
\begin{align*}
  x - y &= 8 \\
  2x - 2y &= 16 \\
  6x - y &= 3
\end{align*}
\]

matrix form

\[
\begin{pmatrix}
  1 & -1 & | & 8 \\
  2 & -2 & | & 16 \\
  6 & -1 & | & 3
\end{pmatrix}
\]

row reduce

\[
\begin{pmatrix}
  1 & 0 & | & -1 \\
  0 & 1 & | & -9 \\
  0 & 0 & | & 0
\end{pmatrix}
\]

solution

\[
\begin{align*}
  x &= -1 \\
  y &= -9
\end{align*}
\]

Conclusion:

\[
\begin{pmatrix}
  1 \\
  2 \\
  6
\end{pmatrix}
- 9
\begin{pmatrix}
  -1 \\
  -2 \\
  -1
\end{pmatrix}
= 
\begin{pmatrix}
  8 \\
  16 \\
  3
\end{pmatrix}
\]

[interactive] ← (this is the picture of a consistent linear system)

What is the relationship between the vectors in the linear combination and the matrix form of the linear equation? They have the same columns!

Shortcut: You can go directly between augmented matrices and vector equations.
The vector equation

\[ x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b, \]

where \( v_1, v_2, \ldots, v_p, b \) are vectors in \( \mathbb{R}^n \) and \( x_1, x_2, \ldots, x_p \) are scalars, has the same solution set as the linear system with augmented matrix

\[
\begin{pmatrix}
| & | & | & |
| v_1 & v_2 & \cdots & v_p |
| & | & | & |
| b |
\end{pmatrix},
\]

where the \( v_i \)'s and \( b \) are the columns of the matrix.

So we now have (at least) two equivalent ways of thinking about linear systems of equations:

1. Augmented matrices.
2. Linear combinations of vectors (vector equations).

The last one is more geometric in nature.
It is important to know what are all linear combinations of a set of vectors \( v_1, v_2, \ldots, v_p \) in \( \mathbb{R}^n \): it’s exactly the collection of all \( b \) in \( \mathbb{R}^n \) such that the vector equation (in the unknowns \( x_1, x_2, \ldots, x_p \))

\[
x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b
\]

has a solution (i.e., is consistent).

**Definition**
Let \( v_1, v_2, \ldots, v_p \) be vectors in \( \mathbb{R}^n \). The span of \( v_1, v_2, \ldots, v_p \) is the collection of all linear combinations of \( v_1, v_2, \ldots, v_p \), and is denoted \( \text{Span}\{v_1, v_2, \ldots, v_p\} \).
In symbols:

\[
\text{Span}\{v_1, v_2, \ldots, v_p\} = \left\{ x_1 v_1 + x_2 v_2 + \cdots + x_p v_p \mid x_1, x_2, \ldots, x_p \text{ in } \mathbb{R} \right\}.
\]

**Synonyms:** \( \text{Span}\{v_1, v_2, \ldots, v_p\} \) is the subset spanned by or generated by \( v_1, v_2, \ldots, v_p \).

This is the first of several definitions in this class that you simply **must learn**. I will give you other ways to think about Span, and ways to draw pictures, but *this is the definition*. Having a vague idea what Span means will not help you solve any exam problems!
Now we have several equivalent ways of making the same statement:

1. A vector $b$ is in the span of $v_1, v_2, \ldots, v_p$.
2. The vector equation

   $$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b$$

   has a solution.
3. The linear system with augmented matrix

   $$\begin{pmatrix}
   v_1 & v_2 & \cdots & v_p & b \\
   \end{pmatrix}$$

   is consistent.

[interactive example]  ←— (this is the picture of an inconsistent linear system)

Note: equivalent means that, for any given list of vectors $v_1, v_2, \ldots, v_p, b$, either all three statements are true, or all three statements are false.
Pictures of Span

Drawing a picture of Span\(\{v_1, v_2, \ldots, v_p\}\) is the same as drawing a picture of all linear combinations of \(v_1, v_2, \ldots, v_p\).

[interactive: span of two vectors in \(\mathbb{R}^2\)]
Pictures of Span

In $\mathbb{R}^3$

[interactive: span of two vectors in $\mathbb{R}^3$]  [interactive: span of three vectors in $\mathbb{R}^3$]
How many vectors are in $\text{Span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$?

A. Zero
B. One
C. Infinity

In general, it appears that $\text{Span}\{v_1, v_2, \ldots, v_p\}$ is the smallest “linear space” (line, plane, etc.) containing the origin and all of the vectors $v_1, v_2, \ldots, v_p$.

We will make this precise later.
The whole lecture was about drawing pictures of systems of linear equations.

- **Points** and **vectors** are two ways of drawing elements of $\mathbb{R}^n$. Vectors are drawn as arrows.
- Vector addition, subtraction, and scalar multiplication have geometric interpretations.
- A **linear combination** is a sum of scalar multiples of vectors. This is also a geometric construction, which leads to lots of pretty pictures.
- The **span** of a set of vectors is the set of all linear combinations of those vectors. It is also fun to draw.
- A system of linear equations is equivalent to a vector equation, where the unknowns are the coefficients of a linear combination.