Math 1553 Worksheet: Chapter 5.1-5.2

1. True or false: If v_1 and v_2 are linearly independent eigenvectors of an $n \times n$ matrix A, then they must correspond to different eigenvalues.

Solution.

False. For example, if $A = I_2$ then e_1 and e_2 are linearly independent eigenvectors both corresponding to the eigenvalue $\lambda = 1$.

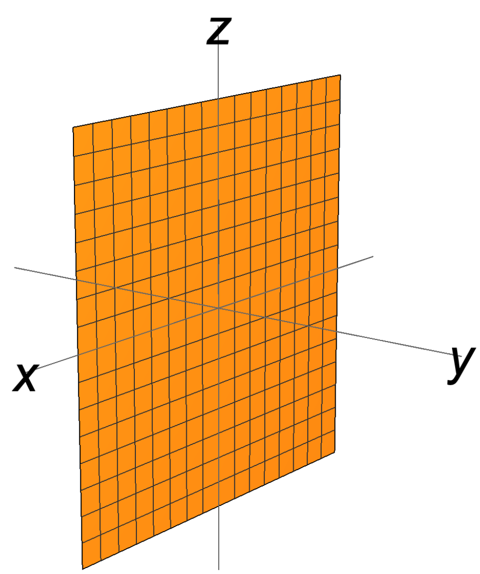
2. In what follows, T is a linear transformation with matrix A. Find the eigenvectors and eigenvalues of A without doing any matrix calculations. (Draw a picture!)

a) $T : \mathbf{R}^3 \to \mathbf{R}^3$ that projects vectors onto the xz-plane in $\mathbf{R}^3$.

b) $T : \mathbf{R}^2 \to \mathbf{R}^2$ that reflects vectors over the line $y = 2x$ in $\mathbf{R}^2$.

Solution.

a) We draw the xz-plane below.

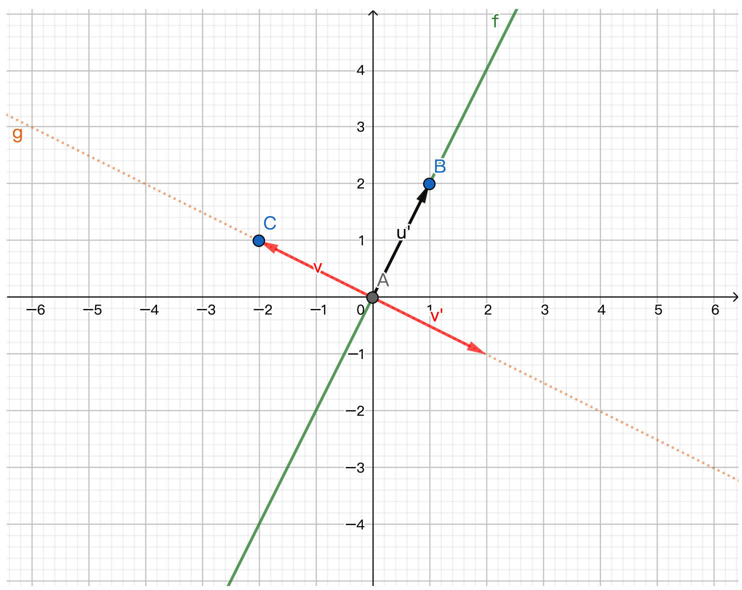


$T(x, y, z) = (x, 0, z)$, so T fixes every vector in the xz-plane and destroys every vector of the form $(0, a, 0)$ with a real. Therefore, $\lambda = 1$ and $\lambda = 0$ are eigenvalues and in fact they are the only eigenvalues since their combined eigenvectors span all of $\mathbf{R}^3$.

The eigenvectors for $\lambda = 1$ are all vectors of the form $\begin{pmatrix} x \\ 0 \\ z \end{pmatrix}$ where at least one of x and z is nonzero, and the eigenvectors for $\lambda = 0$ are all vectors of the form $\begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$ where $y \neq 0$. In other words:

The 1-eigenspace consists of all vectors in Span $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$, while the 0-eigenspace consists of all vectors in Span $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

b) Here is the picture you can play with `https://www.geogebra.org/calculator/xxmhzgev`



T fixes every vector along the line $y = 2x$, so $\lambda = 1$ is an eigenvalue and its eigenvectors are all vectors $\begin{pmatrix} t \\ 2t \end{pmatrix}$ where $t \neq 0$.

T flips every vector along the line perpendicular to $y = 2x$, which is $y = -\frac{1}{2}x$ (for example, $T(-2, 1) = (2, -1)$). Therefore, $\lambda = -1$ is an eigenvalue and its eigenvectors are all vectors of the form $\begin{pmatrix} s \\ -\frac{1}{2}s \end{pmatrix}$ where $s \neq 0$.

3. Answer yes, no, or maybe. Justify your answers. In each case, A is a matrix whose entries are real numbers.

a) Suppose $A = \begin{pmatrix} 3 & 0 & 0 \\ 5 & 1 & 0 \\ -10 & 4 & 7 \end{pmatrix}$. Then the characteristic polynomial of A is

$$\det(A - \lambda I) = (3-\lambda)(1-\lambda)(7-\lambda).$$

b) If A is a 3×3 matrix with characteristic polynomial $-\lambda(\lambda-5)^2$, then the 5-eigenspace is 2-dimensional.

Solution.

a) Yes. Since $A - \lambda I$ is triangular, its determinant is the product of its diagonal entries.

b) Maybe. The geometric multiplicity of $\lambda = 5$ can be 1 or 2. For example, the matrix $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has a 5-eigenspace which is 2-dimensional, whereas the matrix $\begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has a 5-eigenspace which is 1-dimensional. Both matrices have characteristic polynomial $-\lambda(5-\lambda)^2$.

4. Find the eigenvalues and a basis for each eigenspace of $A = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix}$.

Solution.

We solve $0 = \det(A - \lambda I)$.

$$0 = \det\begin{pmatrix} 2-\lambda & 3 & 1 \\ 3 & 2-\lambda & 4 \\ 0 & 0 & -1-\lambda \end{pmatrix} = (-1-\lambda)(-1)^6 \det\begin{pmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{pmatrix} = (-1-\lambda)((2-\lambda)^2 - 9)$$

$$= (-1-\lambda)(\lambda^2 - 4\lambda - 5) = -(\lambda+1)^2(\lambda-5).$$

So $\lambda = -1$ and $\lambda = 5$ are the eigenvalues.

$\underline{\lambda = -1}$: $\left(A + I \mid 0\right) = \left(\begin{array}{ccc|c} 3 & 3 & 1 & 0 \\ 3 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 = R_2 - R_1} \left(\begin{array}{ccc|c} 3 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[\text{then } R_1 = R_1/3]{R_1 = R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$,

with solution $x_1 = -x_2$, $x_2 = x_2$, $x_3 = 0$. The (-1)-eigenspace has basis $\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right\}$.

$\underline{\lambda = 5}$:

$$\left(A - 5I \mid 0\right) = \left(\begin{array}{ccc|c} -3 & 3 & 1 & 0 \\ 3 & -3 & 4 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right) \xrightarrow[R_3 = R_3/(-6)]{R_2 = R_2 + R_1} \left(\begin{array}{ccc|c} -3 & 3 & 1 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow[\text{then } R_2 \longleftrightarrow R_3,\ R_1/(-3)]{R_1 = R_1 - R_3,\ R_2 = R_2 - 5R_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

with solution $x_1 = x_2,\ \ x_2 = x_2,\ \ x_3 = 0$. The 5-eigenspace has basis $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.