Math 1553 Worksheet §3.4

Solutions

1. If A is a 3×5 matrix and B is a 3×2 matrix, which of the following are defined?

a) $A-B$

b) AB

c) A^TB

d) A^2

e) $A+I_5$

f) B^TI_3

Solution.

Only (c) and (f).

a) $A-B$ is nonsense. In order for $A-B$ to be defined, A and B need to have the same number or rows and same number of columns.

b) AB is undefined since the number of columns of A does not equal the number of rows of B.

c) A^T is 5×3 and B is 3×2, so A^TB is a 5×2 matrix.

d) A^2 is nonsense (can't multiply 3×5 with another 3×5).

e) A is 3×5 and I_5 is 5×5. Therefore, $A+I_5$ is not defined.

f) B^T is 2×3 and I_3 is 3×3. Therefore, B^TI_3 is defined (in fact, it is equal to B^T).

2. Suppose A is an $m \times n$ matrix and B is an $n \times m$ matrix. Select all correct answers from the box. It is possible to have more than one correct answer.

a) Suppose x is in $\mathbf{R}^m$. Then ABx *must be* in:
$\boxed{\text{Col}(A), \quad \text{Nul}(A), \quad \text{Col}(B), \quad \text{Nul}(B)}$

b) If $m > n$, then columns of AB could be linearly $\boxed{independent, \quad dependent}$

c) If $m > n$, then columns of BA could be linearly $\boxed{independent, \quad dependent}$

d) If $m > n$ and $Ax = 0$ has nontrivial solutions, then columns of BA could be linearly $\boxed{independent, \quad dependent}$

Solution.

Recall, AB can be computed as A multiplying every column of B. That is $AB = \begin{pmatrix} Ab_1 & Ab_2 & \cdots Ab_m \end{pmatrix}$ where $B = \begin{pmatrix} b_1 & b_2 & \cdots b_m \end{pmatrix}$.

a) $\boxed{\text{Col}(A)}$. Note Bx is a vector in $\mathbf{R}^n$ and $ABx = A(Bx)$ is multiplying A with a vector in $\mathbf{R}^n$. Therefore, ABx is a linear combination of the columns of A, so ABx must be in Col(A).

b) $\boxed{dependent}$. The fact $m > n$ means A has at most n pivots, so $dim(\text{Col}(A)) \leq n$. From part (a) we know that every vector of the form ABx is in Col(A), which has dimension at most n. This means AB can have at most n pivots. But AB is an $m \times m$ matrix and $m > n$, so the columns of AB must be dependent.

c) $\boxed{independent, dependent}$. Both are possible. Since $m > n$, we know that each of A and B can have at most n pivots. The product BA is $n \times n$, so it is possible (though not guaranteed) for BA to have a pivot in each column. For example,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \text{then } BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \text{then } BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

d) $\boxed{dependent}$. Let v be a nontrivial solution to $Ax = 0$. Then v is also a non-trivial solution of $BAx = 0$ since

$$BAv = B(Av) = B(0) = 0.$$

That means $BAx = 0$ has a non-trivial solution, so the columns of BA must be linearly dependent.

In this problem, we made some observations, such as the following.

- Col(AB) is a subset of Col(A).

- Nul(A) is a subset of Nul(BA) since if $Ax = 0$ then $BAx = B(Ax) = B(0) = 0$.

3. True or false. Answer true if the statement is *always* true. Otherwise, answer false.

a) If A, B, and C are nonzero 2×2 matrices satisfying $BA = CA$, then $B = C$.

b) Suppose A is an 4×3 matrix whose associated transformation $T(x) = Ax$ is not one-to-one. Then there must be a 3×3 matrix B which is not the zero matrix and satisfies $AB = 0$.

Solution.

a) False. This question was essentially taken from the "Warnings" slide of the 3.4 PDF slides.

Take $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $BA = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $BC = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, but $B \neq C$.

b) True. If T is not one-to-one then there is a non-zero vector v in $\mathbf{R}^3$ so that

$$Av = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The 3×3 matrix $B = \begin{pmatrix} | & | & | \\ v & v & v \\ | & | & | \end{pmatrix}$ satisfies

$$AB = \begin{pmatrix} | & | & | \\ Av & Av & Av \\ | & | & | \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

4. Consider the following linear transformations:

$T: \mathbf{R}^3 \longrightarrow \mathbf{R}^2$ T projects onto the xy-plane, forgetting the z-coordinate

$U: \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ U rotates clockwise by 90°

$V: \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ V scales the x-direction by a factor of 2.

Let A, B, C be the matrices for T, U, V, respectively.

a) Write A, B, and C.

b) Compute the matrix for $U \circ V \circ T$.

c) Describe U^{-1} and V^{-1}, and compute their matrices.
If you have not yet seen inverse matrices in lecture, describe geometrically the transformation U^{-1} that would "undo" U in the sense that $(U^{-1} \circ U)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$. Now, do the same for V.

Solution.

a) We plug in the unit coordinate vectors:

$$T(e_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad T(e_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad T(e_3) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
$$U(e_1) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad U(e_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
$$V(e_1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad V(e_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \Longrightarrow \quad C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

b) By associativity, we can put the parentheses wherever we wish in computing the product BCA (though we cannot change the order of B, C, and A!):

$$BCA = B(CA) = (BC)A = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix}.$$

c) Intuitively, if we wish to "undo" U, we can imagine that $\begin{pmatrix} x \\ y \end{pmatrix}$. To do this, we need to rotate it 90° *counterclockwise*. Therefore, U^{-1} is counterclockwise rotation by 90°.

Similarly, to undo the transformation V that scales the x-direction by 2, we need to scale the x-direction by $1/2$, so V^{-1} scales the x-direction by a factor of $1/2$.

Their matrices are, respectively,

$$B^{-1} = \frac{1}{0 \cdot 0 - (-1) \cdot 1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$C^{-1} = \frac{1}{2 \cdot 1 - 0 \cdot 0} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}.$$