Math 1553 Reading Day Problem Set, Fall 2021

Solutions

1. T/F: If $\{u, v, w\}$ is a set of linearly dependent vectors, then w must be a linear combination of u and v.

Solution.

False. For example, in $\mathbf{R}^2$ take $u = v = e_1$ and $w = e_2$.

2. Find the value of k that makes the following vectors linearly dependent:

$$\begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -3 \\ k \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}.$$

Solution.

Take the matrix A whose columns are the vectors and row-reduce a bit:

$$A = \begin{pmatrix} -3 & 3 & 3 \\ 0 & -3 & -1 \\ 3 & k & -1 \end{pmatrix} \xrightarrow{R_3 = R_3 + R_1} \begin{pmatrix} -3 & 3 & 3 \\ 0 & -3 & -1 \\ 0 & k+3 & 2 \end{pmatrix}.$$

This matrix will have three pivots unless the second and third rows are multiples of each other. This means the third row is -2 times the second, so $-3(-2) = k + 3$, hence $k = 3$. Alternatively, the student could have computed $\det(A) = 9 - 3k$, so $k = 3$.

3. T/F: If $\{u, v\}$ is a basis for a subspace W, then $\{u - v, u + v\}$ is also a basis for W.

Solution.

True. This was almost directly taken from a Webwork.

4. Which of the following are subspaces of $\mathbf{R}^4$?

(1) The set $W = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \ : 2x - y - z = 0 \right\}$.

(2) The set of solutions to $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & -1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Solution.

(1) is a subspace, since $W = \mathrm{Nul}\begin{pmatrix} 2 & -1 & -1 & 0 \end{pmatrix}$. But (2) is not, since we see immediately that the solution set does not contain the zero vector.

5. T/F: Let W be the set of vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ in $\mathbf{R}^3$ with $abc = 0$. Then W is closed under addition.

Solution.

False. Taken directly from Webwork.

6. Counterclockise rotation by 90°: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Reflection about $y = x$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Clockwise rotation by 90°: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Reflection across the x-axis: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Reflection across the y-axis: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

7. Find k so that the matrix transformation corresponding to the following matrix is not onto:

$$\begin{pmatrix} 1 & 3 & 9 \\ 2 & 6 & k \end{pmatrix}.$$

Solution.

The columns will span $\mathbf{R}^2$ unless they are all collinear, in which case $k = 18$. Alternatively we could find $k = 18$ by row-reducing and determining when the matrix will fail to have a pivot in the second row.

8. Find the **nonzero** value of k so that the matrix is not invertible:

$$\begin{pmatrix} 1 & -1 & 0 \\ k & k^2 & 0 \\ -1 & 1 & 5 \end{pmatrix}.$$

Solution.

We compute the determinant to be $5k(k+1)$, which is zero when $k = -1$. (Note this problem asked for a nonzero value of k, so $k = 0$ is incorrect)

9. Taken almost verbatim from Webwork.
"For each y in $\mathbf{R}^n$, there is at most one x in $\mathbf{R}^m$ so that $T(x) = y$." This means T is $1-1$.

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T is onto.

"For each y in $\mathbf{R}^n$, there is exactly one x in $\mathbf{R}^m$ so that $T(x) = y$." This means T is $1-1$ and onto.

"For each x in $\mathbf{R}^m$, there is exactly one y in $\mathbf{R}^n$ so that $T(x) = y$." This just means T is a transformation.

10. T/F: Suppose A is a 4×6 matrix. Then the dimension of the null space of A is at most 2.

Solution.

False. Just take A to be the 4×6 zero matrix, then its null space has dimension 6.

11. Complete the entries of A so that $\text{Col}(A) = \text{Span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$ and $\text{Nul}(A) = \text{Span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$.

$$A = \begin{pmatrix} r & 1 \\ s & 2 \end{pmatrix}.$$

Solution.

For $\text{Col}(A)$ to be the desired line, $\begin{pmatrix}r\\s\end{pmatrix}$ must be a scalar multiple of $\begin{pmatrix}1\\2\end{pmatrix}$. We also need the null space of A to correspond to the line $x_1 = x_2$, so $-x_1 + x_2 = 0$. Thus $r = -1$ and $s = -2$.

12. Suppose $T : \mathbf{R}^7 \to \mathbf{R}^9$ is a linear transformation with standard matrix A, and suppose the range of T has a basis consisting of 3 vectors. What is $\dim(\text{Nul}A)$?

Solution.

We're given $\dim(\text{range}(T)) = \dim(\text{Col } A) = 3$, and from the domain and codomain of T we know A is a 9×7 matrix, so by the Rank Theorem,

$$\dim(\text{Nul } A) = 7 - \dim(\text{Col } A) = 7 - 3 = 4.$$

13. Define $T : \mathbf{R}^3 \to \mathbf{R}^4$ by

$$T(x, y, z) = (0, x - y, y - x, z).$$

Then T is not onto since, for example, $(1, 0, 0, 0)$ is not in its range. Also, T is not one-to-one since $T(1, 1, 0) = (0, 0, 0, 0)$.
You could also come to the same conclusion by finding the matrix for T and determining it has neither a pivot in each column nor a pivot in each row.

14. Suppose A is a 7×5 matrix, and the null space of A is a line. Say that T is the matrix transformation $T(v) = Av$. What is true about the range of T?

Solution.

Since A is 7×5, we know T has domain $\mathbf{R}^5$ and codomain $\mathbf{R}^7$. Since Nul A is a line we have $\dim(\text{Nul } A) = 1$, so $\dim(\text{Col } A) = 5 - 1 = 4$ by the Rank Theorem.

Since $\text{range}(T) = \text{Col}(A)$, we conclude that the range of T is a 4-dimensional subspace of $\mathbf{R}^7$.

15. Say that $S : \mathbf{R}^2 \to \mathbf{R}^3$ and $T : \mathbf{R}^3 \to \mathbf{R}^4$ are linear transformations. Which of the options given must be true about $T \circ S$?

Solution.

The transformation might be one-to-one. For example if

$$S(x, y) = (x, y, 0) \quad \text{and} \quad T(x, y, z) = (x, y, z, 0), \quad \text{then} \quad (T \circ S)(x, y) = (x, y, 0, 0).$$

However, $T \circ S$ cannot be onto because it corresponds to multiplication by a 4×2 matrix, which has a maximum of 2 pivots and thus cannot have a pivot in each row.

16. T/F: Suppose that A is an invertible $n \times n$ matrix. Then $A + A$ must be invertible.

Solution.

True, since $(2A)^{-1} = \dfrac{1}{2}A^{-1}$.

17. T/F: Suppose A is a 3×3 matrix and the equation $Ax = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ has exactly one solution. Then A must be invertible.

Solution.

True. Since $Ax = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ has exactly one solution, the homogeneous equation $Ax = 0$ has exactly one solution, so A is invertible by the Invertible Matrix Theorem.

18. Suppose that A and B are $n \times n$ matrices and AB is not invertible. Then at least one of A and B must not be invertible: since $0 = \det(AB) = \det(A)\det(B)$, we must have $\det(A) = 0$ or $\det(B) = 0$ (or both).

It is not necessary that A must fail to be invertible (for example, $A = I$ and $B = 0$) and it is not necessary that B must fail to be invertible (for example, $A = 0$ and $B = I$), but at least one of them must not be invertible.

19. Suppose A and B are 3×3 matrices, with $\det(A) = 3$ and $\det(B) = -6$. Find $\det(2A^{-1}B)$.

Solution.

Multiplying a 3×3 matrix by 2 will multiply each row of the matrix by 2, so it will multiply the determinant by 8! We find $\det(A^{-1}B) = -\frac{6}{3} = -2$ so the final answer is -16.

Another way to think about it is the following:

$$\det(2A^{-1}B) = \det(2I * A^{-1} * B) = \det(2I)\det(A^{-1})\det(B) = 8\left(\frac{1}{3}\right)(-6) = -16.$$

20. Let A be the 3×3 matrix satisfying $Ae_1 = e_3$, $Ae_2 = e_2$, and $Ae_3 = 2e_1$. Find $\det(A)$.

Solution.

$A = \begin{pmatrix} Ae_1 & Ae_2 & Ae_3 \end{pmatrix}$, so

$$\det(A) = \det\begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = -2.$$

21. Suppose A is a square matrix and $\lambda = -1$ is an eigenvalue of A. Which of the following must be true?

(1) "A is invertible." Not necessarily, for example $A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$.

(2) "The equation $Ax = x$ has only the trivial solution." Not necessarily. The matrix $A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ satisfies the property that $Ax = x$ has only the trivial solution (i.e. 1 is not an eigenvalue of A), but the matrix $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ satisfies $Ae_2 = e_2$ (i.e. it has 1 as an eigenvalue).

(3) "$\text{Nul}(A+I) = \{0\}$." Not true, in fact $\text{Nul}(A+I)$ must have dimension at least 1.

(4) For some nonzero x, the vectors Ax and x are linearly dependent. True, since $Ax = -x$ and thus Ax is a scalar multiple of x.

(5) "The columns of $A+I$ are linearly independent." Not true, in fact the columns of $A+I$ are linearly dependent.

22. Suppose A is a 4×4 matrix with characteristic polynomial

$$-(1-\lambda)^2(5-\lambda)\lambda.$$

What is the rank of A?

Solution.

We see $\lambda = 0$ is an eigenvalue with algebraic multiplicity 1, so the dimension of the 0-eigenspace must also be 1, which means the null space of A is 1-dimensional. Thus

$$\text{rank}(A) = 4 - 1 = 3.$$

23. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation that reflects across the line $x_2 = 2x_1$. Find the value of k so that $A\begin{pmatrix} 2 \\ k \end{pmatrix} = \begin{pmatrix} 2 \\ k \end{pmatrix}$.

Solution.

A fixes vectors on the line $x_2 = 2x_1$, so we need $\begin{pmatrix} 2 \\ k \end{pmatrix}$ to be on the line $x_2 = 2x_1$, thus $k = 2(2) = 4$.

24. Find the value of k so that $\begin{pmatrix} 1 & k \\ 1 & 3 \end{pmatrix}$ has one real eigenvalue of algebraic multiplicity 2.

Solution.

The characteristic polynomial of the matrix is

$$(1-\lambda)(3-\lambda) - k = \lambda^2 - 4\lambda + 3 - k = (\lambda - 2)^2 - 1 - k.$$

For this matrix have only one real root with algebraic multiplicity 2, it needs to be the perfect square $(\lambda - 2)^2$, so $-1 - k = 0$, thus $k = -1$.

25. Suppose A is a 5×5 matrix with characteristic polynomial

$$(1-\lambda)^3(2-\lambda)(3-\lambda)$$

and that A is diagonalizable. What is the 1-eigenspace of A?

Solution.

Since A is diagonalizable, the algebraic and geometric multiplicities of $\lambda = 1$ must be equal, so the dimension of the 1-eigenspace must be 3.

26. Find the value of t such that 3 is an eigenvalue of $A = \begin{pmatrix} 1 & t & 3 \\ 1 & 1 & 1 \\ 3 & 0 & 3 \end{pmatrix}$.

Solution.

We solve

$$0 = \det(A - 3I) = \begin{pmatrix} -2 & t & 3 \\ 1 & -2 & 1 \\ 3 & 0 & 0 \end{pmatrix} = 3t + 18,$$

so $t = -6$.

27. Suppose A is a 2×2 matrix with characteristic polynomial

$$(1 - \lambda)(2 - \lambda).$$

What is the characteristic polynomial of A^2?

Solution.

From the char. poly. of A we know the eigenvalues of A are 1 and 2. If λ is an eigenvalue of A then λ^2 is an eigenvalue for A^2, since for a λ-eigenvector v of A:

$$A^2 v = A(Av) = A(\lambda v) = \lambda Av = \lambda(\lambda v) = \lambda^2 v.$$

Therefore $1^2 = 1$ and $2^2 = 4$ are eigenvalues of the 2×2 matrix A^2. These are the only possible eigenvalues of A^2 since a 2×2 matrix has at most 2 eigenvalues. Thus the characteristic polynomial of A^2 is

$$(1 - \lambda)(4 - \lambda).$$

28. Suppose that x is an eigenvector of A with eigenvalue 3 and that x is also an eigenvector of B with eigenvalue 4. Then

$$(2A - B)(x) = 2Ax - Bx = 2(3x) - 4x = 2x,$$

so x is an eigenvector for $2A - B$ with corresponding eigenvalue 2.

29. Suppose that A is a 4×4 matrix with eigenvalues 0, 1, and 2, where the eigenvalue 1 has algebraic multiplicity two. Which must be true?
(1) A is not diagonalizable
(2) A is not invertible.

Solution.

Note A might be diagonalizable, for example $A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

However, A is not invertible since it has 0 as an eigenvalue. Therefore, (2) must be true but (1) might not be true.

30. T/F: Suppose A is a 5×5 matrix with real entries. Then A must have at least one real eigenvalue.

Solution.

True, taken almost verbatim from class materials. Complex eigenvalues occur in conjugate pairs, so if A did not have any real eigenvalues then it would need to be $n \times n$ where n is even.

31. Suppose A is a positive stochastic matrix and $A\begin{pmatrix}3/5\\2/5\end{pmatrix} = \begin{pmatrix}3/5\\2/5\end{pmatrix}$, and that $v = \begin{pmatrix}5\\95\end{pmatrix}$. As n gets very large, $A^n v$ approaches what vector $\begin{pmatrix}r\\s\end{pmatrix}$?

Solution.

Since A is positive-stochastic with steady-state vector $w = \begin{pmatrix}3/5\\2/5\end{pmatrix}$, we know that $A^n v$ approaches cw as $n \to \infty$, where c is the sum of entries in v. Thus

$$A^n v \to (5+95)\begin{pmatrix}3/5\\2/5\end{pmatrix} = 100\begin{pmatrix}3/5\\2/5\end{pmatrix} = \begin{pmatrix}60\\40\end{pmatrix}.$$

So $r = 60$ and $s = 40$.

32. Suppose A is a 4×4 matrix of rank 2. Which one of the following must be true?

Solution.

Without ruling out any possibilities, we can observe that (4) below must be true, so it is the answer. For the sake of completeness, we will rule out the other possibilities below anyway.

(1) "A must have four distinct eigenvalues." No. For example, $A = \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$.

(2) "A is not diagonalizable." No. The matrix above is diagonalizable.

(3) "A is diagonalizable." No. Take $A = \begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}$. Then A has rank 2 and its 0-eigenspace has dimension 2 even though $\lambda = 0$ has algebraic multiplicity 3.

(4) "A cannot have four distinct eigenvalues." **True**. The 0-eigenspace (a.k.a. null space) is 2-dimensional, so the algebraic multiplicity of $\lambda = 0$ is at least 2 and therefore A has at most 3 distinct eigenvalues.

33. If A is a 2×2 matrix (with real entries) and A has eigenvalue $1+i$ with corresponding eigenvector $\begin{pmatrix} 2 \\ 1+i \end{pmatrix}$, then from the properties of complex eigenvalues and eigenvectors we know that $1-i$ is an eigenvalue of A with corresponding eigenvector $\begin{pmatrix} 2 \\ 1-i \end{pmatrix}$.

34. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that rotates counterclockwise by 45°, and let A be the standard matrix for T. We know from geometry that Av is never collinear with v if $v \neq 0$, so A has no real eigenvalues. Since the entries of A are real, this means A has two distinct complex eigenvalues.

35. If u and v are orthogonal unit vectors then

$$(3u-8v)\cdot 4u = 3u\cdot 4u - 8v\cdot 4u = 12u\cdot u - 32v\cdot u = 12(1) - 32(0) = 12.$$

36. Find k so that $\begin{pmatrix} 2 \\ k \\ 1 \end{pmatrix}$ and $\begin{pmatrix} k \\ 1 \\ -6 \end{pmatrix}$ are orthogonal.

Solution.

We set $0 = \begin{pmatrix} 2 \\ k \\ 1 \end{pmatrix} \cdot \begin{pmatrix} k \\ 1 \\ -6 \end{pmatrix} = 2k + k - 6$, so $k = 2$.

37. T/F: If W is a subspace of $\mathbf{R}^{100}$ and v is in $W^{\perp}$, then the orthogonal projection of v onto W must be the 0 vector.

Solution.

True: this is a geometric fact but we can also see it through the orthogonal decomposition: since v is in $W^{\perp}$ we know $v = v_{W^{\perp}}$, so the orthogonal decomposition of v with respect to W is $v = v_W + v$, thus $v_W = 0$.

38. T/F: Suppose W is a subspace of $\mathbf{R}^n$. If x is a vector and x_W is the orthogonal projection of x onto W, then $x \cdot x_W = 0$.

Solution.

False. If x is a nonzero vector in W, then $x = x_W$ so $x \cdot x_W = x \cdot x = ||x||^2$. As an example, if W is the x-axis in $\mathbf{R}^2$ and $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then $x_W = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $x \cdot x_W = 1$.

39. Suppose A is a 3×3 invertible matrix. Since $AA^{-1} = I$, the 23-entry of AA^{-1} is 0.

40. Find the orthogonal projection of $\begin{pmatrix}0\\1\end{pmatrix}$ onto Span $\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$.

Solution.

The matrix for the projection is

$$\frac{1}{u\cdot u}uu^T = \frac{1}{1+4}\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}1 & 2\end{pmatrix} = \frac{1}{5}\begin{pmatrix}1 & 2\\2 & 4\end{pmatrix}.$$

Therefore, our answer is

$$\frac{1}{5}\begin{pmatrix}1 & 2\\2 & 4\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2/5\\4/5\end{pmatrix}.$$

41. Consider the orthgonal projection of $\begin{pmatrix}6\\5\\4\end{pmatrix}$ onto Span $\left\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$. What is the first coordinate of the projection?

Solution.

With $A = \begin{pmatrix}1 & 1\\0 & 1\\1 & 0\end{pmatrix}$, we solve $A^TAx = A^T\begin{pmatrix}6\\5\\4\end{pmatrix}$. This gives $\begin{pmatrix}2 & 1\\1 & 2\end{pmatrix}x = \begin{pmatrix}10\\11\end{pmatrix}$, so $x = \begin{pmatrix}3\\4\end{pmatrix}$ and the projection is $Ax = \begin{pmatrix}7\\4\\3\end{pmatrix}$. The first coordinate is 7.

42. Suppose B is the standard matrix for the orthogonal projection of $\mathbf{R}^3$ onto

$$W = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \text{ in } \mathbf{R}^3 \ : \ x+y+2z=0\right\}.$$

What is the dimension of the 1-eigenspace of B?

Solution.

Orthogonal projection onto a subspace fixes all vectors in that subspace, so $Bx = x$ for all x in W. Since $\dim(W) = 2$, the 1-eigenspace of B has dimension 2.

43. Let W be the subspace of $\mathbf{R}^4$ given by all vectors $\begin{pmatrix}x\\y\\z\\w\end{pmatrix}$ satisfying $x+y+z+w=0$.

Find the dimension of $W^\perp$.

Solution.

$W = \text{Nul}\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$, and $\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \end{array}\right)$ has three free variables so $\dim(W) = 3$. If we take a matrix whose columns are a basis for W, then $W^\perp$ is the null space of the 3×4 matrix with those vectors as rows. That matrix will have 3 pivots so its null space will be one-dimensional, thus $\dim(W^\perp) = 1$.

Alternatively, we could use the general observation that the dimensions of any subspace W and its orthogonal complement must add to n if W lives in $\mathbf{R}^n$. Here $n = 4$, so $\dim(W^\perp) = 4 - 3 = 1$.

44. T/F: If b is in Col(A), then every solution to $Ax = b$ is a least-squares solution.

Solution.

True. Taken directly from the 6.5 Webwork.

45. T/F: If A is an $m \times n$ matrix, b is in $\mathbf{R}^m$, and $\widehat{x}$ is a least-squares solution to $Ax = b$, then $\widehat{x}$ is the point in Col(A) that is closest to b.

Solution.

False: $A\widehat{x}$ is the closest point to b in Col A. This one came from the Chapter 6 supplement.

46. Find the least squares solution $\widehat{x}$ to

$$\begin{pmatrix} 6 \\ -2 \\ -2 \end{pmatrix} x = \begin{pmatrix} 14 \\ -2 \\ 0 \end{pmatrix}.$$

Solution.

Here $A = \begin{pmatrix} 6 \\ -2 \\ -2 \end{pmatrix}$ and $b = \begin{pmatrix} 14 \\ -2 \\ 0 \end{pmatrix}$. Solving $A^T A\widehat{x} = A^T b$ gives

$$\begin{pmatrix} 6 & -2 & -2 \end{pmatrix}\begin{pmatrix} 6 \\ -2 \\ -2 \end{pmatrix} x = \begin{pmatrix} 6 & -2 & -2 \end{pmatrix}\begin{pmatrix} 14 \\ -2 \\ 0 \end{pmatrix}, \qquad 44x = 88, \qquad x = 2.$$

47. Find the best fit line $y = Mx + B$ for the data points below using least squares:

$$(-7, -22), \quad (0, -2), \quad (7, 6).$$

Solution.

At first glance it looks long, but the math turns out to be very nice. The system corresponds to

$$-22 = -7M + B, \qquad -2 = 0M + B, \qquad 6 = 7M + B.$$

This system is the system $Ax = b$ below, which is clearly inconsistent because no line goes through all three of the data points.

$$\begin{pmatrix} -7 & 1 \\ 0 & 1 \\ 7 & 1 \end{pmatrix} x = \begin{pmatrix} -22 \\ -2 \\ 6 \end{pmatrix}.$$

Solving $A^T Ax = A^T b$ we get $A^T A = \begin{pmatrix} 98 & 0 \\ 0 & 3 \end{pmatrix}$ and $A^T \begin{pmatrix} -22 \\ -2 \\ 6 \end{pmatrix} = \begin{pmatrix} 196 \\ -18 \end{pmatrix}$. Solving $\left(\begin{array}{cc|c} 98 & 0 & 196 \\ 0 & 3 & -18 \end{array}\right)$ gives $\widehat{x} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$, so the line is $y = 2x - 6$.

48. Let $A = \begin{pmatrix} 4 & 1 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 5 & 2 \end{pmatrix}^{-1}$. Find r and s so that $A^3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} r \\ s \end{pmatrix}$.

Solution.

The matrix A has been diagonalized for us. Note $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ spans the (-1)-eigenspace of A, so $A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$. Repeated multiplication by A just flips $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ back and forth between itself and its negative. Thus $A^2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $A^3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$.

49. T/F: If A is a diagonalizable 6×6 matrix, then A has 6 distinct eigenvalues.

Solution.

False. Just take $A = I_6$. It is diagonalizable (in fact, diagonal!) but it has only one eigenvalue, $\lambda = 1$.

50. Find the eigenvalues of $A = \begin{pmatrix} 1 & 4 \\ 4 & 7 \end{pmatrix}$ and write them in increasing order.

Solution.

$$\det(A - \lambda I) = (1 - \lambda)(7 - \lambda) - 16 = \lambda^2 - 8\lambda - 9 = (\lambda + 1)(\lambda - 9).$$

The eigenvalues of A are the roots of this polynomial, namely $\lambda_1 = -1$ and $\lambda_2 = 9$.