Midterm 2 (2.5-3.4) Solutions

1. Parts (a) and (b) are unrelated. For each statement, answer true or false.

   (a) Let v_1, v_2 and v_3 be vectors in $\mathbf{R}^3$. If $\{v_1, v_2\}$ and $\{v_2, v_3\}$ are linearly independent sets, then $\{v_1, v_2, v_3\}$ must also be linearly independent.

   (b) If $\{w_1, w_2, w_3, w_4\}$ is a linearly dependent set of vectors, then w_4 must be in $\text{Span}\{w_1, w_2, w_3\}$.

   Solution: (a) False. For example, take $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$.

   (b) False. This problem is very similar to one from the 2.5 Webwork, and also the 2.5 Supplement #5. If the set is linearly dependent then at least one of the vectors is in the span of the other vectors, but this doesn't mean that *every* vector is in the span of the other vectors.

2. Find all values of h so that the set of vectors given below is linearly **dependent**.

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ h \\ 4 \end{pmatrix} \right\}$$

   (a) $h = 0$ only

   (b) every real number h except 0

   (c) $h = 2$ only

   (d) $h = 4$ only

   (e) every real number h except 4

   (f) $h = 8$ only

   (g) every real number h except 8

   (h) $h = -8$ only

   (i) every real number h except -8

   Solution: This problem is the 2.5 Webwork #2 with different numbers and a bit of shuffling.

   First, we form a matrix A whose columns are the vectors. Row-reducing quickly gives us

$$A \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & h \\ 0 & 0 & 4 - \frac{h}{2} \end{pmatrix}.$$

   Therefore, the vectors of A will be linearly independent unless $4 - \frac{h}{2} = 0$, which is when $h = 8$.

3. Consider the set $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; ab \leq 0 \right\}$.

   (a) Does V contain the zero vector?

   (b) Is V closed under addition? That is, if u and v are vectors in V, must it be true that $u + v$ is also in V?

   (c) Is V closed under scalar multiplication? That is, if u is in V and c is a real number, must it be true that cu is in V?

Solution: This problem is #3 from Sample Midterm 2A with "≥ 0" replaced by "≤ 0." The answers are the same, and the reasoning is analogous.

(a) Yes. $(0)(0) = 0$ which is less than or equal to 0.

(b) No. For example, $(-1, 0)$ and $(0, -1)$ are in V, but their sum is $(-1, -1)$ which is not in V since $(-1)(-1) > 0$.

(c) Yes. If $u = \begin{pmatrix} a \\ b \end{pmatrix}$ is in V and c is any scalar, then $cu = \begin{pmatrix} ca \\ cb \end{pmatrix}$ which is in V:

$$(ca)(cb) = c^2(ab) \leq 0 \quad \text{since } ab \leq 0.$$

4. Let $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; x_1 + 2x_2 + x_4 = 0 \right\}$. Which one of the following is a basis for V?

(a) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$

(b) $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

(c) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

(d) $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \end{pmatrix} \right\}$

Solution: The answer is (b). This problem is #6 from the 2.6-2.9 Supplement except that the coefficient of x_3 in the 2.6-2.9 Supplement was "-3" and here it is "0."

In this kind of problem, we write V as the null space of a matrix and find a basis for it. Here $V = \text{Nul}(1\ 2\ 0\ 1)$.

We can do the problem in the usual fashion by finding a basis for the null space using parametric vector form, and this will give us the answer (b).

Alternatively, we can just use the process of elimination. From the form of V as Nul$(1\ 2\ 0\ 1)$, we know it has dimension 3, so the only possible answers are (b) and (c). By testing vectors we see that (c) is clearly incorrect, so (b) must be the correct answer.

5. Suppose that A is a 4×2 matrix and that $Ax = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$ has exactly one solution. Which of the following describes the column space of A?

(a) a single point in $\mathbf{R}^2$

(b) a line in $\mathbf{R}^2$

(c) $\mathbf{R}^2$

(d) a single point in $\mathbf{R}^4$

(e) a 2-dimensional plane in $\mathbf{R}^4$

(f) $\mathbf{R}^4$

Solution: The answer is (e). From the given fact that some equation $Ax = b$ exactly one solution, we know that $Ax = 0$ has only the trivial solution, so dim Nul$(A) = 0$. By the rank theorem,

$$2 = \dim \text{Col}(A) + \dim \text{Nul}(A) = \dim \text{Col}(A) + 0,$$

so the column space of A is a plane. Since A is 4×2 the column space lives in $\mathbf{R}^4$.

6. Suppose that A is a 3×3 matrix, that the column space of A is a plane, and that v is a nonzero vector in $\mathbf{R}^3$ satisfying $Av = 0$. Is $\{v\}$ a basis for the null space of A?

(a) Yes, $\{v\}$ must be a basis for the null space of A.

(b) No, $\{v\}$ cannot be a basis for the null space of A.

(c) We need more information to tell whether or not $\{v\}$ is a basis for the null space of A.

Solution: The answer is (a). From the problem and the rank theorem, we know that dim(Nul A)$= 1$, so the null space of A is a line. Therefore, any nonzero vector v in the null space of A will span it, thus $\{v\}$ is a basis for the null space of A.

7. Parts (a) and (b) are unrelated.

(a) Suppose $T : \mathbf{R}^{10} \to \mathbf{R}^8$ is a linear transformation with standard matrix A (so $T(x) = Ax$) and the set of solutions to the equation $Ax = 0$ is 4-dimensional. What is the dimension of the column space of A?

(b) Let $V = \text{Span}\left\{\begin{pmatrix}1\\5\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$. Is $\left\{\begin{pmatrix}2\\10\\0\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix}\right\}$ a basis for V?

Solution: (a) This is very similar to #5a from Sample Midterm 2A. Since A is 8×10, the sum of the dimensions of Nul A and Col A must sum to 10. We're given dim(Nul A)$= 4$, so dim(Col A)$= 6$.

(b) This is a modification of #6 on Sample Midterm 2B. The answer is yes, since the second vectors are also linearly independent and span V.

Notice that $\begin{pmatrix}2\\10\\0\end{pmatrix}$ is just $2\begin{pmatrix}1\\5\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\-1\end{pmatrix}$ is just $-\begin{pmatrix}0\\0\\1\end{pmatrix}$. Therefore, the second set of vectors is a linearly independent subset of 2 vectors in the 2-dimensional subspace V and therefore is a basis for V by the Basis Theorem.

8. Let $A = \begin{pmatrix}2 & 1\\1 & 1\end{pmatrix}$, and let T be the associated matrix transformation $T(x) = Ax$.

(a) Choose the vector v below that satisfies $T(v) = e_1$.

i. $\begin{pmatrix}1\\0\end{pmatrix}$

ii. $\begin{pmatrix}1\\-1\end{pmatrix}$

iii. $\begin{pmatrix}0\\1\end{pmatrix}$

iv. $\begin{pmatrix}2\\1\end{pmatrix}$

(b) How many solutions are there to the equation $T(v) = e_1$?

i. Exactly one solution

ii. Infinitely many solutions

Solution: (a) This is similar to #2 from the 3.1 Webwork, which asked us to solve a different equation of the form $T(x) = b$. Here, we're asked to solve a system of two linear equations in two unknowns. Solving $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ gives the unique solution $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(b) We found in (a) that there is exactly one solution.

9. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$.

Which one of the following is a basis for $\text{Col}(A)$?

(a) $\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}$ (b) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ (c) $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ (d) $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}$

Solution: Without doing any row reduction, we see A has a pivot in each row, so the column span of A is all of $\mathbf{R}^2$, and our answer must be the option that is a basis for $\mathbf{R}^2$. Therefore, (c) is the correct answer. The problem goes slightly beyond the mechanical computation of a basis, as you need to notice that (c) is also a basis for $\mathbf{R}^2$ (just as the pivot columns of the original matrix are also a basis for $\mathbf{R}^2$).

Alternatively, the problem can be done by the process of elimination. Answers (a) and (b) are automatically incorrect because their vectors aren't in $\mathbf{R}^2$, and (d) is automatically incorrect because a set of three vectors in $\mathbf{R}^2$ is linearly dependent and thus can never be a basis for $\mathbf{R}^2$ or any subspace of $\mathbf{R}^2$. This leaves (c) as the only choice.

10. For the transformations (I) through (IV) below, match each transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ with its 2×2 matrix (given by one of the nine options (a) through (i)).

(I) Reflection across the y-axis
(II) Reflection across the line $y = x$
(III) Clockwise rotation by $\pi/2$ radians
(IV) The transformation given by $T(x, y) = (2x - y, x + y)$.

(a) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(f) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(h) $\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$

(i) $\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$

Solution: This problem is a slight modification of #5 from the 3.1 Webwork.
(I) is given by (f)
(II) is given by (c)
(III) is given by (b)
(IV) is given by (h).

11. Suppose $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation.

Answer true or false to each of the following statements.

(a) If $\{v_1 \ldots, v_p\}$ is a linearly independent set of vectors in $\mathbf{R}^n$, then $\{T(v_1), \ldots, T(v_p)\}$ must also be a linearly independent set of vectors.
(b) If $\{v_1 \ldots, v_p\}$ is a linearly independent set of vectors in $\mathbf{R}^n$, then $p \leq n$.

Solution: (a) False. We can't guarantee that $\{T(v_1), \ldots, T(v_p)\}$ is linearly independent unless T is one-to-one.

(b) True. This problem is basically #1a from Sample Midterm 2A. If $p > n$ then any set of p vectors in $\mathbf{R}^n$ is automatically linearly dependent (the matrix with these vectors as its columns will be a "wide" matrix).

12. Let $T(x) = Ax$ where $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & -1 \end{pmatrix}$.

(a) What is the domain of T?

(b) What is the codomain of T?

(c) Which of the following describes the range of T?

(d) Find $T\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Solution: This problem was meant to be a quick concept check of the fundamentals of domain, codomain, and range. A is 3×2, so:
(a) $\mathbf{R}^2$
(b) $\mathbf{R}^3$.
(c) Since A has two linearly independent columns, the range of T is a plane, and this plane lives in $\mathbf{R}^3$ by part (b) of this problem.
(d) $T\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the second column of A, which is $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

13. Which of the following is **not** a condition for a transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ with standard matrix A to be onto?

(a) The columns of A span $\mathbf{R}^m$.

(b) A has a pivot in each row.

(c) The nullspace of A is $\{0\}$.

(d) $Ax = b$ is consistent for all b in $\mathbf{R}^m$.

(e) The range of T has dimension m.

Solution: The answer is (c): if the null space of A is $\{0\}$ then A is one-to-one, but not necessarily onto.

14. Which of the following linear transformations are one-to-one? Select all that apply.

    (a) $T : \mathbf{R}^2 \to \mathbf{R}^2$ that rotates vectors $23°$ counterclockwise.
    (b) $T : \mathbf{R}^2 \to \mathbf{R}^2$ that projects vectors onto the x-axis.
    (c) $T : \mathbf{R}^2 \to \mathbf{R}^3$ given by $T(x_1, x_2) = (x_1 - x_2, x_1 - x_2, x_1)$.
    (d) $T : \mathbf{R}^3 \to \mathbf{R}^3$ given by $T(x) = Ax$, where A is a 3×3 matrix so that $Ax = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ has exactly one solution.

    Solution:
    (a) is one-to-one.
    (b) is not one-to-one because its null space is the entire y-axis.
    (c) is one-to-one because the corresponding matrix has a pivot in each column.
    (d) is one-to-one because the condition for A means that $Ax = 0$ has a unique solution, therefore A must have a pivot in each column.

15. Suppose $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation with standard matrix A, so $T(x) = Ax$.

    Which of the following statements guarantee that T is one-to-one? Select all that apply.

    (a) For every x in $\mathbf{R}^n$, there is at most one y in $\mathbf{R}^m$ so that $T(x) = y$.
    (b) For every y in $\mathbf{R}^m$, there is at most one x in $\mathbf{R}^n$ so that $T(x) = y$.
    (c) Every vector in $\mathbf{R}^m$ is a linear combination of the columns of A.
    (d) Every column in the matrix A is a pivot column.

    Solution: Yes for conditions (b) and (d).
    (a) No, this just means that T might be a transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$.
    (b) Yes, this is the definition of one-to-one.
    (c) No, this means T is onto.
    (d) Yes, this is an equivalent way of saying the matrix transformation $T(x) = Ax$ is one-to-one.

16. Consider the function $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by

    $$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} |y| \\ |x| \end{pmatrix}.$$

    Which of the following properties of a linear transformation does T satisfy?

    (a) $T(cv) = cT(v)$ for all vectors v in $\mathbf{R}^2$ and all real numbers c
    (b) $T(u + v) = T(u) + T(v)$ for all vectors u and v in $\mathbf{R}^2$

    Solution: This problem is a slight modification of problem #1 from the 3.3 Webwork.
    (a) No. For example, if $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then $T(-v)$ does not equal $-T(v)$.
    (b) No. For example, $T \begin{pmatrix} 0 \\ 0 \end{pmatrix} \neq T \begin{pmatrix} 1 \\ 0 \end{pmatrix} + T \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

17. Suppose $T : \mathbf{R}^3 \to \mathbf{R}^3$ is a linear transformation whose range is a plane, and suppose

    $$T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

Which one of the following is a possible standard matrix A for T?

(a) $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

(d) $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & -1 & 0 \end{pmatrix}$

Solution: This is a modification of #11 from Sample Midterm 2B. The answer is (d).

The only options that satisfy $T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ are (c) and (d). However, (d) is the only option whose range is a plane.

18. Suppose $A = \begin{pmatrix} x & y \\ 1 & 2 \end{pmatrix}$.

Find the values of x and y so that $A^2 = A$.

Solution: This is the 3.4 Webwork #6 with some shuffling and changed numbers. Just from setting the 21 entries of A^2 and A equal, we get $2 + x = 1$. Just from setting the 22 entries of A^2 and A equal, we get $4 + y = 2$. Therefore, $x = -1$ and $y = -2$.

(you can compute that with $x = -1$ and $y = -2$, we get $A^2 = A$).

19. For each of the following statements, answer true or false.

(a) If A and B are matrices and the products AB and BA are both defined, then A and B must be square matrices.

(b) Suppose A and B are both $n \times n$ matrices and x is a vector in $\mathbf{R}^n$. If x is in $\text{Nul}(B)$, then x must also be in $\text{Nul}(AB)$.

Solution: (a) False. This problem was taken from #5a in the 3.4 supplemental problems list. If we take A to be 2×3 and B to be 3×2, then the both products AB and BA are defined even though neither A nor B is square.

(b) True. If x is any vector in $\text{Nul}(B)$, then x is also in $\text{Nul}(AB)$ since $ABx = A(Bx) = A(0) = 0$. Similar to #2 from the 3.4 worksheet.

20. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ that first reflects vectors across the line $y = x$, then rotates vectors 90° counterclockwise. Find the standard matrix A for T. In other words, find the matrix A so that $T(v) = Av$.

(a) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

(f) $\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

Solution: This is a slight modification of #7 from the 3.3 Webwork, which also involved finding the standard matrix for a composition involving a reflection and a rotation.

The answer is (a). We could either use matrix multiplication or just find $T(e_1)$ and $T(e_2)$. If we use matrix multiplication then our answer is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is crucial that the matrix for reflection across $y = x$ is the *rightmost* matrix, since it will act on a vector first and then the rotation will happen last.