Final Exam

**1.** Find all solutions to the system of linear equations in $x_1$, $x_2$, and $x_3$ whose augmented matrix is given below. $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right)$

**a)** $x_1 = 1,\ x_2 = 0,\ x_3 = 0$

**b)** $x_1 = 1,\ x_2 = x_2,\ x_3 = x_3$ (where $x_2$ and $x_3$ are any real numbers).

**c)** $x_1 = 1,\ x_2 = 1,\ x_3 = 0$

**d)** $x_1 = 1,\ x_2 = 1,\ x_3 = x_3$ (where $x_3$ is any real number)

**e)** The system is inconsistent, so it has no solutions.

## Solution.

The correct answer is (e). In one step of row reduction, we subtract the second row from the third and get the equation $0 = 1$, so the system is inconsistent.

**2.** Let $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x - 2y - 3z = 0 \right\}$. Which of the following is equal to $W$?

**a)** $\mathrm{Nul}\begin{pmatrix} 1 & -2 & -3 \end{pmatrix}$

**b)** $\mathrm{Col}\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$

**c)** $\mathrm{Row}\begin{pmatrix} 1 & -2 & -3 \end{pmatrix}$

**d)** $\mathrm{Nul}\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$

## Solution.

The correct answer is (a). This is a theme that has come up time after time in the course, starting in chapter 2 and recurring even in chapter 6. The equation that defines $W$ corresponds to the augmented matrix $\left(\begin{array}{ccc|c} 1 & -2 & -3 & 0 \end{array}\right)$, so $W$ is $\mathrm{Nul}\begin{pmatrix} 1 & -2 & -3 \end{pmatrix}$.

**3.** Consider the set

$$S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \mid x + y \geq 0 \right\}.$$

Below, determine which properties of a subspace are satisfied by $S$.

**a)** Does $S$ contain the zero vector?

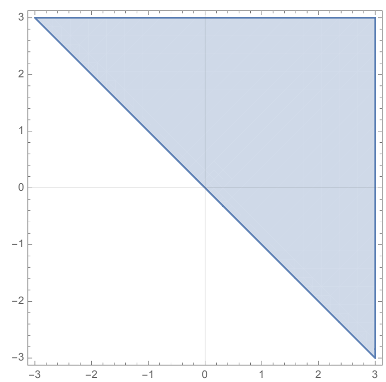
**b)** Is $S$ closed under addition? That is, if $u$ and $v$ are vectors in $S$, must it be true that $u+v$ is also in $S$?

**c)** Is $S$ closed under scalar multiplication? That is, if $c$ is a real number and $u$ is in $S$, must it be true that $cu$ is in $V$?

## Solution.

This is #2 from the Practice Final exam, with one "−" sign changed to a "+" sign.

One way to do this problem is to draw the region $S$, which is given by $x \geq -y$. It is the triangular region shaded in blue below.




**a)** Yes, since $0+0=0 \geq 0$.

**b)** Yes. If $u = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $v = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ are in $S$, then for $u+v = \begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix}$ we have

$$(x_1+x_2)+(y_1+y_2) = (x_1+y_1)+(x_2+y_2).$$

Both $x_1+y_1$ and $x_2+y_2$ are nonnegative, so their sum is too.

**c)** No. For example, $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is in $S$, but $-u = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ is not in $S$ since $-1-1<0$.

**4.** Answer yes or no to each of the following questions.

**a)** Suppose that $A$ is a $5 \times 7$ matrix of rank 5 and that $u$ and $v$ are vectors so that Span$\{u, v\}$ is the null space of $A$. Must it be true that $u$ and $v$ are linearly independent?

**b)** Suppose that $A$ is a $7 \times 5$ matrix and that the null space of $A$ consists of only the zero vector. Suppose that $b$ is a vector in $\mathbf{R}^7$. Must it be true that $Ax = b$ has exactly one solution?

## Solution.

**a)** Yes. By the rank theorem,

$$\dim(\text{Nul } A) = 7 - \dim(\text{Col } A) = 7-5 = 2,$$

so the null space of $A$ is 2-dimensional. Therefore, $u$ and $v$ are vectors so that $\text{Span}\{u, v\}$ is a plane, so $u$ and $v$ must be linearly independent.

**b)** No. Since $\dim(\text{Col } A) = 5$ and $\text{Col } A$ lives in $\mathbf{R}^7$, there will be (infinitely many) vectors $b$ in $\mathbf{R}^7$ so that $Ax = b$ is inconsistent.

**5.** Suppose that $T : \mathbf{R}^3 \to \mathbf{R}^3$ is the linear transformation of orthogonal projection onto a two-dimensional plane that contains the origin. Let $A$ be the standard matrix for $T$, so $T(x) = Ax$ for all $x$ in $\mathbf{R}^3$.

**a)** What is the dimension of the null space of $A$?

**b)** Is $A$ diagonalizable?

## Solution.

**a)** The answer is 1. Since the range of $T$ is a plane, we have $\dim(\text{Col } A) = 2$, so $\dim(\text{Nul } A) = 3 - 2 = 1$.

**b)** Yes, since $A$ is the matrix for an orthogonal projection, it must be diagonalizable (standard fact from chapter 6).

**6.** Consider the matrix $A$ and its reduced row echelon form (RREF) given below:

$$A = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 0 & 2 & -1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}.$$

**a)** Which of the following vectors is in $\text{Nul}(A)$?

(i) $\begin{pmatrix} -1/2 \\ -1/2 \\ 0 \end{pmatrix}$ (ii) $\begin{pmatrix} 1/2 \\ 1/2 \\ 1 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 \\ 1 \\ 1/2 \end{pmatrix}$ (iv) $\begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$ (v) $\begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}$

**b)** Is $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ in the column space of $A$?

## Solution.

This is a classic problem testing fundamental knowledge of $\text{Col } A$ and $\text{Nul } A$ (for example, see the 2.5-3.1 worksheet #4).

**a)** The answer is (ii). From the RREF of $A$ we see that vectors $\text{Nul } A$ satisfy

$$x_1 - \frac{1}{2}x_3 = 0 \quad \text{and} \quad x_2 - \frac{1}{2}x_3 = 0.$$

In parametric form, this is $x_3 \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \end{pmatrix}$.

**b)** True. In fact, $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ is the second column of $A$!

**7.** Answer true or false to each of the following questions.

**a)** The set $\left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \right\}$ is linearly independent.

**b)** Suppose $u$ and $v$ are linearly independent vectors in $\mathbf{R}^3$. Then $\{u,\ v,\ u-v\}$ must be linearly independent.

## Solution.

**a)** True, no row-reduction required. If we let $A$ be the matrix whose columns are those three vectors, it is immediate that $A$ has a pivot in every column, so the set is linearly independent.

**b)** False: the third vector in the set is $u-v$, which is in Span$\{u,v\}$.

**8.** Which of the following transformations are linear? Select all that apply.

**a)** The transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ defined by $T(x_1,x_2,x_3) = (x_1,\ x_2,\ 1)$.

**b)** The transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x_1,x_2,x_3) = (x_1-2x_2-3x_3,\ 0)$.

**c)** The transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x,y) = (x+5,\ y+5)$.

**d)** The transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x,y) = (\sin(x),\ y)$.

## Solution.

**a)** No: for example $T(0,0,0) = (0,0,1)$, so $T$ does not send the zero vector to the zero vector.

**b)** Yes, and in fact $T(x) = \begin{pmatrix} 1 & -2 & -3 \\ 0 & 0 & 0 \end{pmatrix} x$.

**c)** No: for example $T(0,0) = (5,5)$, so the same reasoning as in part (a) applies.

**d)** No: $\sin(x)$ ruins it. For example,

$$T(\pi,0) = (0,0) \quad \text{but} \quad 2T(\pi/2,0) = 2(1,0) = (2,0).$$

**9.** For the transformations (I) through (IV) below, match each transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ with its $2\times 2$ matrix (given by one of the nine options (a) through (i)).

(I) Reflection across the $x$-axis
(II) Counterclockwise rotation by $\pi/4$ radians
(III) Reflection across the line $y = x$
(IV) The transformation given by $T(x,y) = (-y,-x)$.

**a)** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

**b)** $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

**c)** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

**d)** $\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

**e)** $\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

**f)** $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

**g)** $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

**h)** $\begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix}$

**i)** $\begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}$

**Solution.**

This problem is #10 from Midterm 2 with very small modifications.

(I) Reflection across the $x$-axis: answer (g) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(II) Counterclockwise rotation by $\pi/4$ radians: answer (e) $\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

(III) Reflection across the line $y = x$: answer (a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(IV) The transformation given by $T(x, y) = (-y, -x)$: answer (b) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

**10.** Suppose $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation with standard matrix $A$.
Answer true or false to each of the following questions.

**a)** If $T$ is onto, then the columns of $A$ must span $\mathbf{R}^m$.

**b)** If $T$ is one-to-one, then the columns of $A$ must be linearly independent.

**Solution.**

**a)** True. Follows directly from the definition of onto.

**b)** True. Follows directly from the definition of one-to-one.

**11.** Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation given by $T(x_1, x_2) = (x_1 - x_2, x_2)$, and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that rotates vectors by 90° **clockwise**. Find the standard matrix $C$ for $T \circ U$. In other words, find the matrix $C$ so that

$$(T \circ U)(x) = Cx \quad \text{for all } x \text{ in } \mathbf{R}^2.$$

**a)** $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$

**b)** $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$

**c)** $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$

**d)** $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$

**e)** $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$

## Solution.

The correct answer is (e). This is #20 from midterm 2 with slight modifications to the transformations $T$ and $U$.

$$(T \circ U)(x) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(x) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}x.$$

**12.** If $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 3$, find the determinant of the matrix below.

$$\begin{pmatrix} d & e & f \\ g & h & i \\ a-2g & b-2h & c-2i \end{pmatrix}$$

**a)** 0

**b)** $-3$

**c)** 3

**d)** $-6$

**e)** 6

## Solution.

The answer is 3. We've seen this kind of problem many times before: #4 from Midterm 3, the Determinants I Webwork #7, Quiz 6, and a problem from Sample Midterm 3.

To get from the first matrix to the second, we need to do $R_1 \longleftrightarrow R_2$, then $R_2 \longleftrightarrow R_3$, then subtract $2R_2$ from $R_3$. The first two operations will multiply the determinant by $-1$ twice, while the last operation is a row replacement which does not affect the determinant. Therefore,

$$\begin{pmatrix} d & e & f \\ g & h & i \\ a-2g & b-2h & c-2i \end{pmatrix} = 3(-1)(-1)(1) = 3.$$

**13.** Suppose $A$ and $B$ are $2 \times 2$ matrices satisfying $\det(A) = 4$ and $\det(B) = 8$. Find $\det(-4AB^{-1})$.

**a)** 2

**b)** $-2$

**c)** 4

**d)** $-4$

**e)** 8

**f)** $-8$

**g)** 16

**h)** $-16$

## Solution.

The answer is 8. This was taken from #16 on the Practice Final and is nearly identical to #19 from the Reading Day list.

$$\det(-4AB^{-1}) = (-4)^2 \det(A)\det(B^{-1}) = 16(4)(1/8) = 8.$$

**14.** Find the value of $c$ so that the matrix below has exactly one real eigenvalue with algebraic multiplicity 2.

$$A = \begin{pmatrix} 2 & c \\ -5 & 12 \end{pmatrix}.$$

## Solution.

The answer is 5. This is #21 from the Practice Final (also #24 from the Reading Day list, and #12 from Midterm 3) with numbers changed. The characteristic polynomial of $A$ is

$$\det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & c \\ -5 & 12-\lambda \end{pmatrix} = \lambda^2 - 14\lambda + 24 + 5c.$$

For this to be a perfect square it must be $(\lambda-7)^2 = \lambda^2 - 14\lambda + 49$, so $24 + 5c = 49$, thus $c = 5$.

**15.** Suppose $A$ is a $3 \times 3$ matrix with characteristic polynomial

$$\det(A - \lambda I) = (2-\lambda)^2(-1-\lambda).$$

**a)** Is $A$ invertible?

**b)** Is $A$ diagonalizable?

## Solution.

**a)** Yes. From the characteristic polynomial of $A$ we see its eigenvalues are 2 and $-1$, so 0 is not an eigenvalue of $A$ and therefore $A$ is invertible. Or alternatively, we can compute

$$\det(A) = \det(A - 0I) = (2-0)^2(-1-0) \neq 0,$$

so $A$ is invertible.

**b)** Maybe. It depends on whether the eigenvalue $\lambda = 2$ has geometric multiplicity 1 or 2. We give examples of each case below. The matrix on the left is diagonalizable (in fact, diagonal!) but the matrix on the right is not diagonalizable since $\mathrm{Nul}(A - 2I)$ is only one-dimensional.

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

**16.** Find a basis for the 3-eigenspace of $A = \begin{pmatrix} 4 & 2 & 4 \\ 1 & 5 & 4 \\ 1 & 0 & 3 \end{pmatrix}$.

**a)** $\left\{ \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$

**b)** $\left\{ \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\}$

**c)** $\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$

**d)** $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$

**e)** $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\}$

## Solution.

The answer is (b).

We compute $\left( A - 3I \mid 0 \right) = \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 1 & 2 & 4 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) \longrightarrow RREF \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$.

Thus $x = 0$ and $y = -2z$, so parametric form gives

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -2z \\ z \end{pmatrix} = z\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}.$$

**17.** Find the inverse of the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

**a)** $\begin{pmatrix} -1 & 3 \\ 2 & -4 \end{pmatrix}$

**b)** $\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$

**c)** $\begin{pmatrix} \frac{1}{2} & -\frac{3}{2} \\ -1 & 2 \end{pmatrix}$

**d)** $\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$

## Solution.

The answer is (d). This is #1 from Midterm 3 with changed numbers.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \frac{1}{4-6}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

**18.** Answer yes, no, or maybe to each of the following questions.

**a)** Suppose that $A$ is a $5 \times 5$ matrix and that the set of solutions to $Ax = e_1$ is a line in $\mathbf{R}^5$. Is $A$ invertible?

**b)** Suppose that $A$ is a $3 \times 3$ matrix whose 1-eigenspace is a line in $\mathbf{R}^3$ and whose 4-eigenspace is a plane in $\mathbf{R}^3$. Is $A$ diagonalizable?

## Solution.

**a)** No. This was taken verbatim from #2b from Midterm 3.

**b)** Yes. This was taken nearly verbatim from #2a from Midterm 3.

**19.** Answer each of the following questions.

**a)** Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation for orthogonal projection onto the line $y = 2x$, and let $A$ be the standard matrix for $T$. What are the eigenvalues of $A$?
(i) 1 only
(ii) 0 and 1
(iii) $-1$ and 1
(iv) 0 and 2

(v) 1 and 2
(vi) 0 only

**b)** Let $B$ be the $2\times 2$ matrix that reflects vectors in $\mathbf{R}^2$ across the line spanned by $\begin{pmatrix}-2\\3\end{pmatrix}$. Which vector $v$ below is in the $(-1)$-eigenspace of $B$?

(i) $v=\begin{pmatrix}0\\-3\end{pmatrix}$

(ii) $v=\begin{pmatrix}-2\\3\end{pmatrix}$

(iii) $v=\begin{pmatrix}-3\\2\end{pmatrix}$

(iv) $v=\begin{pmatrix}-2\\0\end{pmatrix}$

(v) $v=\begin{pmatrix}3\\2\end{pmatrix}$

## Solution.

**a)** The answer is (ii). Since $A$ is the matrix for an orthogonal projection and $A$ is not the identity matrix or zero matrix, its eigenvalues are 0 and 1 (standard fact from 6.3).

**b)** The answer is (v). This is nearly #20 on the Practice Final with changed numbers. As we have seen many times with $2\times 2$ reflection matrices, the $(-1)$-eigenspace of $B$ is the line perpendicular to $\mathrm{Span}\left\{\begin{pmatrix}-2\\3\end{pmatrix}\right\}$. This line is $\mathrm{Span}\left\{\begin{pmatrix}3\\2\end{pmatrix}\right\}$.

**20.** Answer yes or no to each the following questions.

**a)** It is possible for $\lambda = 0$ to be an eigenvalue of an $n\times n$ matrix.

**b)** It is possible for the zero vector to be an eigenvector of an $n\times n$ matrix.

## Solution.

**a)** Yes. Any $n\times n$ matrix which is not invertible will have 0 as an eigenvalue.

**b)** No. By definition of eigenvector, the zero vector is never an eigenvector.

**21.** Let $A$ be the matrix whose diagonalization $A = CDC^{-1}$ is given below.

$$A=\begin{pmatrix}2&3\\4&1\end{pmatrix}\begin{pmatrix}-1&0\\0&2\end{pmatrix}\begin{pmatrix}2&3\\4&1\end{pmatrix}^{-1}.$$

Which of the following statements are true? Select all that apply.

**a)** $A^2\begin{pmatrix}2\\4\end{pmatrix}=\begin{pmatrix}2\\4\end{pmatrix}$.

**b)** Another diagonalization of $A$ is

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}^{-1}.$$

**c)** The eigenvalues of $A$ are $-1$ and 2.

**d)** The matrix equation $(A - I)x = 0$ has only the trivial solution.

## Solution.

**a)** True. This is an easier version of #15b from Midterm 3. The diagonalization says that $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ is an eigenvector for eigenvalue $-1$, so $A^2 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = (-1)^2 \begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

**b)** False. This second attempt at diagonalizing $A$ changes the order of the eigenvectors without changing the order of the eigenvalues appropriately.

**c)** True, these are the diagonal entries of $\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$.

**d)** True, since $\lambda = 1$ is not an eigenvalue of $A$.

**22.** Which of the following matrices are diagonalizable? Select all that apply.

**a)** $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

**b)** $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

**c)** $C = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

**d)** $D = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

## Solution.

This is #16 from Midterm 3 with some changed numbers.

**a)** No, $A$ is not diagonalizable. Its only eigenvalue is $\lambda = 2$ and the 2-eigenspace is a line.

**b)** Yes: $B$ is $2 \times 2$ with the two different eigenvalues 1 and $-1$.

**c)** Yes: $C$ is $2 \times 2$ with the two different eigenvalues 1 and $-1$. Here $C$ was triangular, so no work as necessary to find the eigenvalues.

**d)** Yes: $D$ is is $2 \times 2$ with the two different eigenvalues 0 and 2.

**23.** Let $A$ be the $2\times 2$ matrix that has $\begin{pmatrix}2\\1\end{pmatrix}$ as eigenvector corresponding to the eigenvalue 2, and $\begin{pmatrix}0\\1\end{pmatrix}$ as eigenvector corresponding to the eigenvalue $-1$.

Find $A\begin{pmatrix}4\\1\end{pmatrix}$ and enter its two entries below.

**Solution.**

We write $\begin{pmatrix}4\\1\end{pmatrix}$ as a linear combination of the two eigenvectors:

$$\begin{pmatrix}4\\1\end{pmatrix}=2\begin{pmatrix}2\\1\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}.$$

Therefore,

$$A\begin{pmatrix}4\\1\end{pmatrix}=2A\begin{pmatrix}2\\1\end{pmatrix}-A\begin{pmatrix}0\\1\end{pmatrix}=2\cdot 2\begin{pmatrix}2\\1\end{pmatrix}-\begin{pmatrix}0\\-1\end{pmatrix}=\begin{pmatrix}8\\4\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}8\\5\end{pmatrix}.$$

**24.**

**a)** Suppose $A$ is a $7\times 7$ matrix whose entries are real numbers. Then $A$ must have at least one real eigenvalue.

**b)** The matrix $A=\begin{pmatrix}5 & 1\\ -5 & 3\end{pmatrix}$ has $\lambda = 4+2i$ as an eigenvalue. Which of the following is the other eigenvalue of $A$?
(i) $4-2i$
(ii) $-4+2i$
(iii) $2+4i$
(iv) $2-4i$

**Solution.**

**a)** True. Taken from #4 in the 5.5 Webwork.

**b)** The answer is (i), since the other eigenvalue is $\overline{4+2i}=4-2i$.

* This ends midterm 3 problems

**25.** This problem has two unrelated parts.

**a)** Suppose $A$ is an $n\times n$ positive stochastic matrix.
Is the 1-eigenspace of $A$ one-dimensional?

**b)** The matrix $B=\begin{pmatrix}1/12 & 5/12\\ 11/12 & 7/12\end{pmatrix}$ has the property that

$$\mathrm{Nul}(B-I)=\mathrm{Span}\left\{\begin{pmatrix}5\\11\end{pmatrix}\right\}.$$

What vector does $B^n \begin{pmatrix} 24 \\ 8 \end{pmatrix}$ approach as $n$ gets very large?

(i) $\begin{pmatrix} 24/16 \\ 8/16 \end{pmatrix}$

(ii) $\begin{pmatrix} 24 \\ 8 \end{pmatrix}$

(iii) $\begin{pmatrix} 5 \\ 11 \end{pmatrix}$

(iv) $\begin{pmatrix} 120 \\ 88 \end{pmatrix}$

(v) $\begin{pmatrix} 10 \\ 22 \end{pmatrix}$

(vi) $\begin{pmatrix} 120/11 \\ 8 \end{pmatrix}$

## Solution.

**a)** True, by the Perron-Frobenius Theorem.

**b)** The correct answer is (ii). $B$ is positive stochastic and the steady state vector is $\begin{pmatrix} 5/16 \\ 11/16 \end{pmatrix}$, so

$$B^n \begin{pmatrix} 24 \\ 8 \end{pmatrix} \longrightarrow 32 \begin{pmatrix} 5/16 \\ 11/16 \end{pmatrix} = \begin{pmatrix} 10 \\ 22 \end{pmatrix}.$$

This problem could also be done by the process of elimination. The only choices whose entries sum to 32 are (ii) and (v), but (v) is clearly false due to the given 1-eigenvector, so (ii) must be the answer.

**26.** Find the steady state vector for the matrix $A = \begin{pmatrix} 1/4 & 3/4 \\ 3/4 & 1/4 \end{pmatrix}$.

**a)** $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

**b)** $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$

**c)** $\begin{pmatrix} 1/4 \\ 3/4 \end{pmatrix}$

**d)** $\begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix}$

**e)** $\begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}$

**f)** $\begin{pmatrix} -1/2 \\ 3/2 \end{pmatrix}$

**Solution.**

We see $A$ is positive stochastic. $\left(A-I \mid 0\right)=\left(\begin{array}{cc|c} -3/4 & 3/4 & 0 \\ 3/4 & -3/4 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$, so the 1-eigenspace is spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus, $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ is the steady-state vector.

**27.** Answer true or false to each of the following questions.

**a)** Suppose $u$, $v$, and $w$ are vectors in $\mathbf{R}^n$. If $u$ is orthogonal to $v$ and $u$ is orthogonal to $w$, then $u$ must be orthogonal to every vector in $\mathrm{Span}\{v,w\}$.

**b)** Suppose $v_1$ and $v_2$ are nonzero vectors in $\mathbf{R}^n$ and $v_1 \cdot v_2 = 0$. Then $\{v_1, v_2\}$ must be linearly independent.

**Solution.**

**a)** True, a fundamental fact from chapter 6 from the discussion about orthogonal complements. If $u \cdot v = 0$ and $u \cdot w = 0$, then for any scalars $c_1$ and $c_2$, we get the following by using properties of dot products:

$$u \cdot (c_1 v + c_2 w) = c_1(u \cdot v) + c_2(u \cdot w) = c_1(0) + c_2(0) = 0,$$

so $u$ is orthogonal to all vectors in $\mathrm{Span}\{v,w\}$.

**b)** True. This is #33a from the Practice Final exam.

**28.** Suppose $W = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$, and let $P$ be the matrix for orthogonal projection onto $W$. Which of the following are true? Select all that apply.

**a)** $P^2 = P$.

**b)** The 1-eigenspace of $P$ is 2-dimensional.

**c)** $\mathrm{Nul}(P) = W^\perp$.

**d)** $P\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

**Solution.**

**a)** True. Standard fact about orthogonal projection matrices.

**b)** True. The 1-eigenspace of $P$ is $W$, which has dimension 2.

**c)** True. Standard fact about orthogonal projection matrices.

**d)** True: $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ is in $W^{\perp}$ since it is orthogonal to both basis vectors given for $W$, so $P\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

**29.** Suppose $u$ and $v$ are vectors in $\mathbf{R}^4$ satisfying $||u|| = 2$, $||v|| = 3$, and $u \cdot v = 3$. Find $||3u - v||^2$.

### Solution.

This is a slight modification of #28 from the Practice Final.

$$\begin{aligned} ||3u - v||^2 &= (3u - v) \cdot (3u - v) = 9(u \cdot u) - 3(u \cdot v) - 3(v \cdot u) + v \cdot v \\ &= 9(2^2) - 3(3) - 3(3) + 3^2 = 27. \end{aligned}$$

**30.** Suppose that $W$ is a subspace of $\mathbf{R}^3$ and $x$ is the vector whose orthogonal decomposition is $x = x_W + x_{W^\perp}$, where

$$x_W = \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} \quad \text{and} \quad x_{W^\perp} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}.$$

**a)** What is the closest vector to $x$ in $W$?

(i) $\begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix}$ (ii) $\begin{pmatrix} -6 \\ 1 \\ -1 \end{pmatrix}$ (iii) $\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$ (iv) $\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$ (v) $\begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}$

**b)** What is the distance from $x$ to $W$?

(i) $\sqrt{7}$ (ii) 7 (iii) 3 (iv) $\sqrt{21}$ (v) 17 (vi) $\sqrt{17}$

### Solution.

This problem is #38 from the Practice Final with new numbers.

**a)** The answer is (v). The closest vector to $x$ in $W$ is $x_W = \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}$.

**b)** The answer is (vi). The distance from $x$ to $W$ is

$$||x_{W^\perp}|| = \sqrt{2^2 + (-2)^2 + 3^2} = \sqrt{17}.$$

**31.** Let $W$ be the span of $\begin{pmatrix}1\\1\\4\end{pmatrix}$. Which of the following is a basis of $W^\perp$?

**a)** $\left\{\begin{pmatrix}-2\\-2\\1\end{pmatrix}\right\}$

**b)** $\left\{\begin{pmatrix}1\\1\\-4\end{pmatrix}\right\}$

**c)** $\left\{\begin{pmatrix}1 & 1 & 4\end{pmatrix}\right\}$

**d)** $\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}-4\\0\\1\end{pmatrix}\right\}$

**e)** $\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}4\\0\\1\end{pmatrix}\right\}$

## Solution.

The answer is (d): $W^\perp = \mathrm{Nul}\begin{pmatrix}1 & 1 & 4\end{pmatrix}$, and a standard parametric vector form computation gives us a basis for $W^\perp$ consisting of $\begin{pmatrix}-1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}-4\\0\\1\end{pmatrix}$.

Alternatively, we could have done this using a concept check and the process of elimination. Since $W$ is a 1-dimensional subspace of $\mathbf{R}^3$ we know $W^\perp$ is 2-dimensional, so the only possible answers are (d) and (e). Choice (e) is clearly wrong, since neither of its vectors is orthogonal to $W$.

**32.** Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation of orthogonal projection onto the line $y = -x$. Find the standard matrix $A$ for $T$. In other words, find the matrix $A$ so $T(x) = Ax$ for all $x$ in $\mathbf{R}^2$.

**a)** $A = \begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}$

**b)** $A = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$

**c)** $A = \begin{pmatrix}1/2 & 0\\0 & -1/2\end{pmatrix}$

**d)** $A = \begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$

**e)** $A = \begin{pmatrix}0 & -1/2\\-1/2 & 0\end{pmatrix}$

**f)** $A = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$

**g)** $A = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$

**h)** $A = \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$

### Solution.

The answer is (g). This problem is basically #34 from the Practice Final, using a different line through the origin.

We use the formula $\dfrac{1}{u \cdot u} u u^T$, where $u$ is **any** nonzero vector along the line $y = -x$.

For example, we could take $u = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$$A = \frac{1}{u \cdot u} u u^T = \frac{1}{1+1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

**33.** Answer true or false to each of the following questions.

**a)** Suppose that $W$ is a subspace of $\mathbf{R}^4$, $A$ is a $4 \times 3$ matrix so that $\mathrm{Col}(A) = W$, and $b$ is a vector in $\mathbf{R}^4$. If $\widehat{x}$ is a least-squares solution to $Ax = b$, then $\widehat{x}$ is the point in $W$ that is closest to $b$.

**b)** If $y$ is a vector in a subspace $W$ of $\mathbf{R}^n$, then the orthogonal projection of $y$ onto $W$ is $y$.

### Solution.

**a)** False. $A\widehat{x}$ is the point in $W$ closest to $b$. This is almost identical to #45 on the Practice Exam and also to Chapter 6 Supplement 1(d).

**b)** True. Taken verbatim from #4 of the 6.3 Webwork.

**34.** Let $b = \begin{pmatrix} 4 \\ 6 \\ 10 \end{pmatrix}$ and $W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$.
Find the orthogonal projection of $b$ onto $W$.

### Solution.

The answer is $\begin{pmatrix} 7 \\ 6 \\ 7 \end{pmatrix}$. With $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}$, we solve $A^T A v = A^T b$ for $v$.

$$A^T A = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}, \qquad A^T b = \begin{pmatrix} 14 \\ 8 \end{pmatrix}.$$

Solving $\left(\begin{array}{cc|c} 2 & 2 & 14 \\ 2 & 3 & 8 \end{array}\right)$ gives $v = \begin{pmatrix} 13 \\ -6 \end{pmatrix}$, so our answer is

$$Av = \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 13 \\ -6 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \\ 7 \end{pmatrix}.$$

**35.** Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Find the least-squares solution $\widehat{x}$ to the equation $Ax = b$. Which of the following vectors is $\widehat{x}$?

(i) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ (iv) $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

(v) $\begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$ (vi) $\begin{pmatrix} 1/6 \\ 2/6 \\ 3/6 \end{pmatrix}$ (vii) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

### Solution.

The answer is (i). We solve $A^T A\widehat{x} = A^T b$.

$$A^T A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A^T b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \text{which gives} \quad \widehat{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Alternatively, we can do this problem just by understanding least-squares. Since $\mathrm{Col}(A)$ is the $xy$-plane of $\mathbf{R}^3$, we conclude with no work that $b_{\mathrm{Col}\ A} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, and it is clear that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, so $\widehat{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

**36.** The goal of this problem is to find $A$ and $b$ that will enable us to find the least-squares line $y = Mx + B$ that best fits the data $(-2, 1)$, $(1, 2)$, and $(5, -1)$.

In other words, we need $A$ and $b$ that will enable us to find the least-squares solution to $A\begin{pmatrix} M \\ B \end{pmatrix} = b$.

**a)** What is the matrix $A$ in the equation $A\begin{pmatrix} M \\ B \end{pmatrix} = b$?

(i) $A = \begin{pmatrix} -2 & 1 \\ 1 & 1 \\ 5 & 1 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ -1 & 1 \end{pmatrix}$ (iii) $A = \begin{pmatrix} 1 & -2 \\ 1 & 1 \\ 1 & 5 \end{pmatrix}$

(iv) $A = \begin{pmatrix} -2 & 1 & 5 \\ 1 & 1 & 1 \end{pmatrix}$ (v) $A = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 1 & 5 \end{pmatrix}$ (vi) $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & -1 \end{pmatrix}$

**b)** What is the vector $b$ in the equation $A\begin{pmatrix} M \\ B \end{pmatrix} = b$?

(i) $b = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$ (ii) $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ (iii) $b = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ (iv) $b = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

(v) $b = \begin{pmatrix} 1 & 2 & -1 \end{pmatrix}$ (vi) $b = \begin{pmatrix} -2 & 1 & 5 \end{pmatrix}$

## Solution.

For this problem, we copied #37 from the Practice Exam and then changed the data points. It is otherwise identical.

We do both parts of this problem together. Note that the problem specifies the order of $M$ and $B$ where $y = Mx + B$ is the least-squares line, so we need to be careful with the ordering of our columns or we will mix up $M$ and $B$.

$$\begin{aligned} x = -2, y = 1: &\quad 1 = M(-2) + B \\ x = 1, y = 2: &\quad 2 = M(1) + B \\ x = 5, y = -1: &\quad -1 = M(5) + B \end{aligned}$$

This gives us the system

$$\begin{aligned} -2M + B &= 1 \\ M + B &= 2 \\ 5M + B &= -1 \end{aligned}$$

which corresponds to the matrix equation

$$\begin{pmatrix} -2 & 1 \\ 1 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} M \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

Therefore, $A = \begin{pmatrix} -2 & 1 \\ 1 & 1 \\ 5 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

The answer to (a) is (i), and the answer to (b) is (iv).