Section 6.2

Orthogonal Complements

Orthogonal Complements

Definition

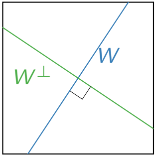
Let W be a subspace of $\mathbf{R}^n$. Its **orthogonal complement** is

$$W^{\perp} = \{ v \text{ in } \mathbf{R}^n \mid v \cdot w = 0 \text{ for all } w \text{ in } W \} \qquad \text{read "}W\text{ perp".}$$

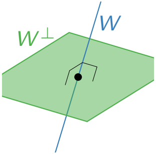
$W^{\perp}$ is orthogonal complement
A^T is transpose

Pictures:

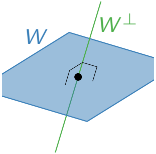
The orthogonal complement of a line in $\mathbf{R}^2$ is the perpendicular line. [interactive]



The orthogonal complement of a line in $\mathbf{R}^3$ is the perpendicular plane. [interactive]



The orthogonal complement of a plane in $\mathbf{R}^3$ is the perpendicular line. [interactive]



Poll

Let W be a subspace of $\mathbf{R}^n$.

Facts:

1. $W^\perp$ is also a subspace of $\mathbf{R}^n$
2. $(W^\perp)^\perp = W$
3. $\dim W + \dim W^\perp = n$
4. If $A = \begin{pmatrix} v_1 & v_2 & \cdots & v_m \end{pmatrix}$ and $W = \operatorname{Col} A$, then $W^\perp = \operatorname{Nul}(A^T)$ since

$$\begin{aligned} W^\perp &= \text{all vectors orthogonal to each } v_1, v_2, \ldots, v_m \\ &= \left\{x \text{ in } \mathbf{R}^n \mid x \cdot v_i = 0 \text{ for all } i = 1, 2, \ldots, m\right\} \\ &= \operatorname{Nul}\begin{pmatrix} — v_1^T — \\ — v_2^T — \\ \vdots \\ — v_m^T — \end{pmatrix} = \operatorname{Nul}(A^T). \end{aligned}$$

Problem: if $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$, compute $W^\perp$.

[interactive]

$$\text{Span}\{v_1, v_2, \ldots, v_m\}^\perp = \text{Nul}\begin{pmatrix} \text{---}\; v_1^T \;\text{---} \\ \text{---}\; v_2^T \;\text{---} \\ \vdots \\ \text{---}\; v_m^T \;\text{---} \end{pmatrix}$$

Definition

The **row space** of an $m \times n$ matrix A is the span of the *rows* of A. It is denoted Row A. Equivalently, it is the column space of A^T:

$$\text{Row}\, A = \text{Col}\, A^T.$$

It is a subspace of $\mathbf{R}^n$.

We showed before that if A has rows $v_1^T, v_2^T, \ldots, v_m^T$, then

$$\text{Span}\{v_1, v_2, \ldots, v_m\}^\perp = \text{Nul}\, A.$$

Hence we have shown:

Fact: $(\text{Row}\, A)^\perp = \text{Nul}\, A$.

Replacing A by A^T, and remembering $\text{Row}\, A^T = \text{Col}\, A$:

Fact: $(\text{Col}\, A)^\perp = \text{Nul}\, A^T$.

Using property 2 and taking the orthogonal complements of both sides, we get:

Fact: $(\text{Nul}\, A)^\perp = \text{Row}\, A$ and $\text{Col}\, A = (\text{Nul}\, A^T)^\perp$.

Dimension of the row space

Even though Row(A) lives in $\mathbf{R}^n$ and Col(A) lives in $\mathbf{R}^m$ if A is an $m \times n$ matrix, both subspaces have the same dimension.

Theorem

If A is an $m \times n$ matrix, then $\dim(\text{Row } A) = \dim(\text{Col}\, A)$.

Orthogonal Complements of Most of the Subspaces We've Seen

For any vectors $v_1, v_2, \ldots, v_m$:

$$\mathrm{Span}\{v_1, v_2, \ldots, v_m\}^\perp = \mathrm{Nul}\begin{pmatrix} — v_1^T — \\ — v_2^T — \\ \vdots \\ — v_m^T — \end{pmatrix}$$

For any matrix A:

$$\mathrm{Row}\,A = \mathrm{Col}\,A^T$$

and

$$(\mathrm{Row}\,A)^\perp = \mathrm{Nul}\,A \qquad \mathrm{Row}\,A = (\mathrm{Nul}\,A)^\perp$$

$$(\mathrm{Col}\,A)^\perp = \mathrm{Nul}\,A^T \qquad \mathrm{Col}\,A = (\mathrm{Nul}\,A^T)^\perp$$

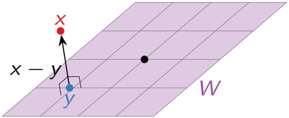
For any other subspace W, first find a basis $v_1, \ldots, v_m$, then use the above trick to compute $W^\perp = \mathrm{Span}\{v_1, \ldots, v_m\}^\perp$.

Section 6.3

Orthogonal Projections (will finish in next set of slides)

Best Approximation

Suppose you measure a data point x which you know for theoretical reasons must lie on a subspace W.



Due to measurement error, though, the measured x is not actually in W. Best approximation: y is the *closest* point to x on W.

How do you know that y is the closest point? The vector from y to x is orthogonal to W: it is in the *orthogonal complement* $W^{\perp}$.

Orthogonal Decomposition

Theorem

Every vector x in $\mathbf{R}^n$ can be written as

$$x = x_W + x_{W^\perp}$$

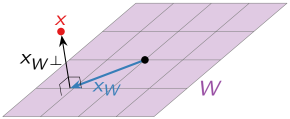
for unique vectors x_W in W and $x_{W^\perp}$ in $W^\perp$.

The equation $x = x_W + x_{W^\perp}$ is called the **orthogonal decomposition** of x (with respect to W).

The vector x_W is the **orthogonal projection** of x onto W.

The vector x_W *is the closest vector to* x *on* W.

[interactive 1] [interactive 2]



Theorem

Every vector x in $\mathbf{R}^n$ can be written as

$$x = x_W + x_{W^\perp}$$

for unique vectors x_W in W and $x_{W^\perp}$ in $W^\perp$.

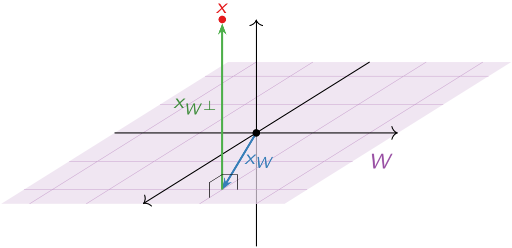
Why?

Let W be the xy-plane in $\mathbf{R}^3$. Then $W^\perp$ is the z-axis.

$$x = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \implies x_W = \qquad\qquad x_{W^\perp} = \qquad .$$

$$x = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \implies x_W = \qquad\qquad x_{W^\perp} = \qquad .$$

This is just decomposing a vector into a "horizontal" component (in the xy-plane) and a "vertical" component (on the z-axis).



[interactive]

Problem: Given x and W, how do you compute the decomposition $x = x_W + x_{W^\perp}$?

Observation: It is enough to compute x_W, because $x_{W^\perp} = x - x_W$.

The $A^T A$ Trick

Theorem (The $A^T A$ Trick)

Let W be a subspace of $\mathbf{R}^n$, let $v_1, v_2, \ldots, v_m$ be a spanning set for W (e.g., a basis), and let

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{pmatrix}.$$

Then for any x in $\mathbf{R}^n$, the matrix equation

$$A^T A v = A^T x \quad \text{(in the unknown vector } v\text{)}$$

is consistent, and $x_W = Av$ for any solution v.

Recipe for Computing $x = x_W + x_{W^\perp}$

- Write W as a column space of a matrix A.
- Find a solution v of $A^T A v = A^T x$ (by row reducing).
- Then $x_W = Av$ and $x_{W^\perp} = x - x_W$.

Problem: Compute the orthogonal projection of a vector $x = (x_1, x_2, x_3)$ in $\mathbf{R}^3$ onto the xy-plane.

Problem: Let

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x_1 - x_2 + x_3 = 0 \right\}.$$

Compute the distance from x to W.

Problem: Let

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x_1 - x_2 + x_3 = 0 \right\}.$$

Compute the distance from x to W.

[interactive]

Theorem (The $A^T A$ Trick)

Let W be a subspace of $\mathbf{R}^n$, let $v_1, v_2, \ldots, v_m$ be a spanning set for W (e.g., a basis), and let

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{pmatrix}.$$

Then for any x in $\mathbf{R}^n$, the matrix equation

$$A^T A v = A^T x \qquad \text{(in the unknown vector } v\text{)}$$

is consistent, and $x_W = Av$ for any solution v.

Proof:

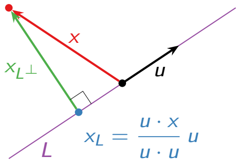
Orthogonal Projection onto a Line

Problem: Let $L = \mathrm{Span}\{u\}$ be a line in $\mathbf{R}^n$ and let x be a vector in $\mathbf{R}^n$. Compute x_L.

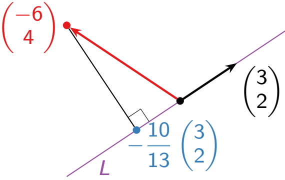
Projection onto a Line

The projection of x onto a line $L = \mathrm{Span}\{u\}$ is

$$x_L = \frac{u \cdot x}{u \cdot u}\, u \qquad x_{L^\perp} = x - x_L.$$



Problem: Compute the orthogonal projection of $x = \binom{-6}{4}$ onto the line L spanned by $u = \binom{3}{2}$, and find the distance from u to L.



[interactive]

Summary

Let W be a subspace of $\mathbf{R}^n$.

- The **orthogonal complement** $W^\perp$ is the set of all vectors orthogonal to everything in W.
- We have $(W^\perp)^\perp = W$ and $\dim W + \dim W^\perp = n$.
- $\text{Row}\, A = \text{Col}\, A^T$, $(\text{Row}\, A)^\perp = \text{Nul}\, A$, $\text{Row}\, A = (\text{Nul}\, A)^\perp$, $(\text{Col}\, A)^\perp = \text{Nul}\, A^T$, $\text{Col}\, A = (\text{Nul}\, A^T)^\perp$.
- **Orthogonal decomposition:** any vector x in $\mathbf{R}^n$ can be written in a unique way as $x = x_W + x_{W^\perp}$ for x_W in W and $x_{W^\perp}$ in $W^\perp$. The vector x_W is the **orthogonal projection** of x onto W.
- The vector x_W is the *closest point to x in W*: it is the *best approximation*.
- The *distance* from x to W is $\|x_{W^\perp}\|$.
- If $W = \text{Col}\, A$ then to compute x_W, solve the equation $A^T A v = A^T x$; then $x_W = Av$.
- If $W = L = \text{Span}\{u\}$ is a line then $x_L = \frac{u \cdot x}{u \cdot u}\, u$.