Math 1553 Worksheet §§5.1, 5.2, 5.4

1. Answer yes, no, or maybe. Justify your answers. In each case, A is a matrix whose entries are real numbers.

a) Suppose $A = \begin{pmatrix} 3 & 0 & 0 \\ 5 & 1 & 0 \\ -10 & 4 & 7 \end{pmatrix}$. Then the characteristic polynomial of A is

$$\det(A - \lambda I) = (3-\lambda)(1-\lambda)(7-\lambda).$$

b) If A is a 3×3 matrix with characteristic polynomial $-\lambda(\lambda-5)^2$, then the 5-eigenspace is 2-dimensional.

c) If A is an invertible 2×2 matrix, then A is diagonalizable.

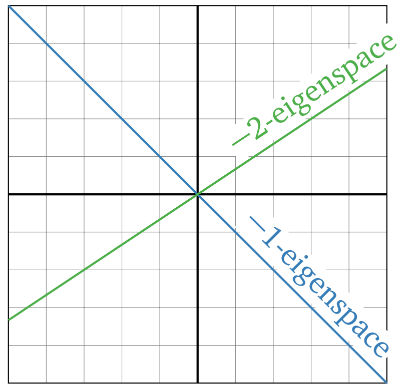
Solution.

a) Yes. Since $A - \lambda I$ is triangular, its determinant is the product of its diagonal entries.

b) Maybe. The geometric multiplicity of $\lambda = 5$ can be 1 or 2. For example, the matrix $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has a 5-eigenspace which is 2-dimensional, whereas the matrix $\begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has a 5-eigenspace which is 1-dimensional. Both matrices have characteristic polynomial $-\lambda(5-\lambda)^2$.

c) Maybe. The identity matrix is invertible and diagonalizable, but the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is invertible but not diagonalizable.

2. The eigenspaces of some 2×2 matrix A are drawn below. Write an invertible matrix C and a diagonal matrix D so that $A = CDC^{-1}$.



Solution: We choose D to be a diagonal matrix whose entries are the eigenvalues of A, and C a matrix whose columns are corresponding eigenvectors (written in the same order).

The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = -2$.
The (-1)-eigenspace is spanned by $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
The (-2)-eigenspace is spanned by $v_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.
Therefore, we can choose $C = \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$.

There are other possibilities for C and D.
For example, since $\text{Span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\} = \text{Span}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$, we could have chosen $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ instead. Regardless, if you write any correct answers for C and D and go the extra step of carrying out the computation, you will obtain

$$A = CDC^{-1} = -\frac{1}{5}\begin{pmatrix} 8 & 3 \\ 2 & 7 \end{pmatrix}.$$

3. Let

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1}.$$

Find a formula for A^n (where n is a positive integer).

Solution: The matrix A has already been diagonalized for us as $A = CDC^{-1}$ for the matrices above. We find $C^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ so

$$\begin{aligned} A^n = CD^nC^{-1} &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2^n} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2^n} & \frac{1}{2^n} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 1 - \frac{1}{2^n} & \frac{1}{2^n} \end{pmatrix}. \end{aligned}$$