Math 1553 Worksheet §§3.4-3.6

Solutions

1. True or false (justify your answer). Answer true if the statement is *always* true. Otherwise, answer false.

a) If A is a 3×4 matrix and B is a 4×2 matrix, then the linear transformation Z defined by $Z(x) = ABx$ has domain $\mathbf{R}^3$ and codomain $\mathbf{R}^2$.

b) If A is an $n \times n$ matrix and the equation $Ax = b$ has at least one solution for each b in $\mathbf{R}^n$, then the solution is *unique* for each b in $\mathbf{R}^n$.

c) Suppose A is an $n \times n$ matrix and every vector in $\mathbf{R}^n$ can be written as a linear combination of the columns of A. Then A must be invertible.

Solution.

a) False. In order for Bx to make sense, x must be in $\mathbf{R}^2$, and so Bx is in $\mathbf{R}^4$ and $A(Bx)$ is in $\mathbf{R}^3$. Therefore, the domain of Z is $\mathbf{R}^2$ and the codomain of Z is $\mathbf{R}^3$.

b) True. The first part says the transformation $T(x) = Ax$ is onto. Since A is $n \times n$, this is the same as saying A is invertible, so T is one-to-one and onto. Therefore, the equation $Ax = b$ has exactly one solution for each b in $\mathbf{R}^n$.

c) True. If the columns of A span $\mathbf{R}^n$, then A is invertible by the Invertible Matrix Theorem. We can also see this directly without quoting the IMT:

If the columns of A span $\mathbf{R}^n$, then A has n pivots, so A has a pivot in each row and column, hence its matrix transformation $T(x) = Ax$ is one-to-one and onto and thus invertible. Therefore, A is invertible.

2. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be rotation *clockwise* by $60°$. Let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation satisfying $U(1,0) = (-2,1)$ and $U(0,1) = (1,0)$.

a) Find the standard matrix for the composition $U \circ T$ using matrix multiplication.

b) Find the standard matrix for the composition $T \circ U$ using matrix multiplication.

c) Is rotating clockwise by $60°$ and then performing U, the same as first performing U and then rotating clockwise by $60°$?

Solution.

a) The matrix for T is $\begin{pmatrix} \cos(-60°) & -\sin(-60°) \\ \sin(-60°) & \cos(-60°) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$.

The matrix for U is $\begin{pmatrix} U(e_1) & U(e_2) \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$.

The matrix for $U \circ T$ is

$$\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -1 - \frac{\sqrt{3}}{2} & \frac{1}{2} - \sqrt{3} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

b) The matrix for $T \circ U$ is

$$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 + \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} + \sqrt{3} & -\frac{\sqrt{3}}{2} \end{pmatrix}.$$

c) No. In (a) and (b), we found that the standard matrices for $U \circ T$ and $T \circ U$ are different, so the transformations are different.