Math 1553 Worksheet §2.6, 2.7, 2.9, 3.1

Solutions

1. Circle **TRUE** if the statement is always true, and circle **FALSE** otherwise.

a) If A is a 3×100 matrix of rank 2, then $\dim(\mathrm{Nul}A) = 97$.

TRUE **FALSE**

b) If A is an $m \times n$ matrix and $Ax = 0$ has only the trivial solution, then the columns of A form a basis for $\mathbf{R}^m$.

TRUE **FALSE**

c) The set $V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; x - 4z = 0 \right\}$ is a subspace of $\mathbf{R}^4$.

TRUE **FALSE**

Solution.

a) False. By the Rank Theorem, $\mathrm{rank}(A) + \dim(\mathrm{Nul}A) = 100$, so $\dim(\mathrm{Nul}A) = 98$.

b) False. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ has only the trivial solution for $Ax = 0$, but its column space is a 2-dimensional subspace of $\mathbf{R}^3$.

c) True. V is $\mathrm{Nul}(A)$ for the 1×4 matrix A below, and therefore is automatically a subspace of $\mathbf{R}^4$:

$$A = \begin{pmatrix} 1 & 0 & -4 & 0 \end{pmatrix}.$$

Alternatively, we could verify the subspace properties directly if we wished, but this is much more work!

(1) The zero vector is in V, since $0 - 4(0)0 = 0$.

(2) Let $u = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix}$ and $v = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix}$ be in V, so $x_1 - 4z_1 = 0$ and $x_2 - 4z_2 = 0$. We compute

$$u + v = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \\ w_1 + w_2 \end{pmatrix}.$$

Is $(x_1 + x_2) - 4(z_1 + z_2) = 0$? Yes, since

$$(x_1 + x_2) - 4(z_1 + z_2) = (x_1 - 4z_1) \; + \; (x_2 - 4z_2) = 0 + 0 = 0.$$

(3) If $u = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ is in V then so is cu for any scalar c:

$$cu = \begin{pmatrix} cx \\ cy \\ cz \\ cw \end{pmatrix} \quad \text{and} \quad cx - 4cz = c(x - 4z) = c(0) = 0.$$

2. Write a matrix A so that $\operatorname{Col} A = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}\right\}$ and $\operatorname{Nul} A$ is the xz-plane.

Solution.

Many examples are possible. We'd like to design an A with the prescribed column span, so that $\left(A \mid 0\right)$ will have free variables x_1 and x_3. One way to do this is simply to leave the x_1 and x_3 columns blank, and make the second column $\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$. This guarantees that A destroys the xz-plane and has the column span required.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

An alternative method for finding the same matrix: Write $A = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$. We want the column span to be the span of $\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$ and we want

$$A\begin{pmatrix} x \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}\begin{pmatrix} x \\ 0 \\ z \end{pmatrix} = xv_1 + zv_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{for all } x \text{ and } z.$$

One way to do this is choose $v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, and $v_2 = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$.

3. Let $A = \begin{pmatrix} 1 & -5 & -2 & -4 \\ 2 & 3 & 9 & 5 \\ 1 & 1 & 4 & 2 \end{pmatrix}$, and let T be the matrix transformation associated to A, so $T(x) = Ax$.

a) What is the domain of T? What is the codomain of T? Give an example of a vector in the range of T.

b) The RREF of A is $\begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Is there a vector in the codomain of T which is not in the range of T? Justify your answer.

Solution.

a) The domain is $\mathbf{R}^4$; the codomain is $\mathbf{R}^3$. The vector $0 = T(0)$ is contained in the range, as is

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = T \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

b) Yes. The range of T is the column span of A, and from the RREF of A we know A only has two pivots, so its column span is a 2-dimensional subspace of $\mathbf{R}^3$. Since $\dim(\mathbf{R}^3) = 3$, the range is not equal to $\mathbf{R}^3$.