Math 1553 Midterm exam 3 solutions

1.

2. For all $n \times n$ matrices A and B we have $\det(A+B) = \det(A) + \det(B)$.

Solution.

False. det is a very complex nonlinear operator. Linearity does not hold in general. For example Let $A = I, B = -I$ then $\det(I - I) = 0 \neq 1 + (-1)^n$ whenever n is even.

3. A matrix and its reduced row echelon form must have the same eigenvalues.

Solution.

False. $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ has RREF $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

4. A is $n \times n$ matrix, then the dimensions of the eigenspaces for A must add up to n

Solution.

False. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has only 1-dimensional eigenspace.

5. Say that λ is a real number and that A is a square matrix. Then λ is an eigenvalue for A if and only if $A - \lambda I$ is invertible.

Solution.

False. "if and only if $A - \lambda I$ is singular (not invertible)" would be correct.

6. A matrix A is diagonalizable and A invertible, then A^{-1} is diagonalizable.

Solution.

True. $A = PDP^{-1}$ Then $A^{-1} = (PDP^{-1})^{-1} = PD^{-1}P^{-1}$

7. Complete the following definition. Say that A is an $n \times n$ matrix, v is a vector in $\mathbb{R}^n$ and λ is a real number. Then v is an eigenvector for A with eigenvalue λ if..

Solution.

if $Av = \lambda v$ and $v \neq 0$

8. Say that A is 3×3 matrix and that there are non-zero vectors x, y and z with $Ax = x$, $Ay = -2y$, $Az = 0$.

Which of the following statements must be true? Select only one answer.

Solution.

A is diagonalizable but is not invertible. The eigenvalues of A are $1, -2, 0$ so it is not invertible. All eigenvalues are distinct therefore, there are 3 linearly independent eigenvectors to diagonalize A.

9. Suppose that A and B are 2×2 matrices satisfying $\det A = 2$ and $A^2B = \begin{pmatrix} 1 & 3 \\ 2 & 18 \end{pmatrix}$. What is the determinant of B?

Solution.

False. $\det(A^2B) = 18 - 6 = 12$ and also $= (\det A)^2 \det B = 4 \det B$. So $\det B = 3$

10. Find the determinant of the following matrix.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix}$$

Solution.

0. Since the matrix has rows are the same.

11. Suppose that the determinant of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is 1. What is the determinant of $\begin{pmatrix} 2c & 2d \\ 2a & 2b \end{pmatrix}$

Solution.

Use three row operations on A: $R_1 \longleftrightarrow R_2$, $2 \times R_1$, $2 \times R_2$. Therefore, there are 3 changes: $\times(-1), \times 2, \times 2$. So the determinant is -4.

12. Suppose that S is an oval in $\mathbf{R}^2$ with area 7. What is the area of $T(S)$, if $T : \mathbf{R}^2 \to \mathbf{R}^2$ is the linear transformation with standard matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$?

Solution.

Area of $T(S)$ is $|\det A| \times area(S) = |4 - 6| \times 7 = 14$.

13. Find the nonzero value of a that makes the matrix $\begin{pmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 2 & a & a \end{pmatrix}$ not invertible.

(Note: the question is asking for a nonzero value of a, so 0 is not a correct answer)

Solution.

Do row reductions and try to take it to REF.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 2 & a & a \end{pmatrix} \xrightarrow{R_3-2R_1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & a & a-2 \end{pmatrix} \xrightarrow{R_3-R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & a-2 \end{pmatrix}$$

If we select $a = 0$ or 2, then the matrix is not invertible. Since it ask for nonzero solutions, so it must be $a = 2$.

14. A is $n \times n$ matrix with n distinct real eigenvalues, and we choose one eigenvector for each eigenvalue, then the chosen eigenvectors must be linearly independent.

Solution.

True. This is a theorem.

15. Find the value of a so that the following matrix has one real eigenvalue of algebraic multiplicity 2. $A = \begin{pmatrix} 1 & a \\ -1 & -3 \end{pmatrix}$

Solution.

First compute characteristic polynomial

$$\det(A - \lambda I) = (1-\lambda)(-3-\lambda) + a = \lambda^2 + 2\lambda - 3 + a = (\lambda+1)^2 - 4 + a$$

. Since we want algebraic multiplicity 2, we must have only one unique root for the polynomial, therefore $a = 4$.

16. A is 4×4 matrix with eigenvalues 7, 8, and 9. Suppose we also know that the 7-eigenspace is 2-dimensional. Can we conclude that A is diagonalizable?

Solution.

Yes, it is diagonalizable. It means (geometric multiplicity) G.M(7)=2. Then we know G.M.(8)$\geq$ 1, G.M.(9)$\geq$ 1. Therefore we have at least 4 independent eigenvectors.

17. Suppose A is 2×2 matrix whose entries are real numbers, and suppose that $\lambda = 3 + 2i$ as an eigenvalue of A with corresponding eigenvector $\begin{pmatrix} -2 \\ -2i \end{pmatrix}$. Which one of the following statements must be true?

Solution.

$3 - 2i$ is an eigenvalue of A, with corresponding eigenvector $\begin{pmatrix} -2 \\ 2i \end{pmatrix}$. Just take complex conjugate.

18. Select the 2×2 matrix A whose 33-eigenspace is spanned by $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and whose (-1)-eigenspace is spanned by $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Solution.

$$A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}^{-1}$$

19. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation given by $T(x, y, z) = (x, y, 0)$ and let A be the standard matrix for T. Which one of the following statements must be true?

Solution.

"The eigenvalue $\lambda = 1$ has algebraic multiplicity 2 and the 1-eigenspace has dimension 2“. The transformation is projection onto $xy - plane$. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

20. Let $A = \begin{pmatrix} 2 & 2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 3 & -4 \end{pmatrix}^{-1}$. Find the number c so that $A\begin{pmatrix} 2 \\ c \end{pmatrix} = 3\begin{pmatrix} 2 \\ c \end{pmatrix}$.

Solution.

$\begin{pmatrix} 2 \\ c \end{pmatrix}$ is an eigenvector corresponding to $\lambda = 3$, so $c = -4$.

21. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation of rotation counterclockwise by 30 degrees, and let A be the standard matrix for T. Which one of the following statements must be true about A?

Solution.

"A has two distinct complex eigenvalues". $A = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$ has complex eigenvalues.

22. Suppose A is a positive stochastic 2×2 matrix and $A\begin{pmatrix} \frac{3}{7} \\ \frac{4}{7} \end{pmatrix} = \begin{pmatrix} \frac{3}{7} \\ \frac{4}{7} \end{pmatrix}$.

As n gets large, $A^n\begin{pmatrix} 10 \\ 4 \end{pmatrix}$ approaches the vector $v = \begin{pmatrix} a \\ 8 \end{pmatrix}$. Find a.

Solution.

We know v must be a multiple of steady-state vector $\begin{pmatrix} \frac{3}{7} \\ \frac{4}{7} \end{pmatrix}$, so $a = 6$.

23. Consider an internet with 3 pages. Page 1 links to pages 2 and 3. Page 2 links only to page 1. Page 3 links only to page 2.

What is the importance matrix A (also known as the Google matrix) for this internet?

Solution.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{pmatrix}$$