MATH 1553, SPRING 2020
MIDTERM 2, LECTURE C1-C4 (11:15 AM - 12:05 PM)

Name	C Key	Section	

Please **read all instructions** carefully before beginning.

- Write your name on the front of each page (not just the cover page!).
- The maximum score on this exam is 50 points, and you have 50 minutes to complete this exam.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- As always, $e_1, e_2, \ldots, e_n$ refer to the standard unit coordinate vectors of $\mathbf{R}^n$.
- Show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit! If you cannot fit your work on the front side of the page, use the back side of the page and indicate that you are using the back side.
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All work must be written on the exam itself.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

Problem 1. [10 points]

True or false. Circle **T** if the statement is *always* true.
Otherwise, circle **F**. You do not need to show work or justify your answer.

a) **T** **F** Suppose v_1, v_2, v_3, v_4 are vectors in $\mathbf{R}^4$ and

$$\text{Span}\{v_1, v_2, v_3, v_4\} = \mathbf{R}^4.$$

Then $\{v_1, v_2, v_3, v_4\}$ must be a basis for $\mathbf{R}^4$.

Basis Theorem

b) **T** **F** Suppose A is a 3×3 matrix and the equation $Ax = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ has infinitely many solutions. Then A is not invertible.

$A\vec{x} = \vec{0}$ has inf. many sol's

$\Rightarrow$ A not invertible

c) **T** **F** If a matrix A has more rows than columns, then the matrix transformation $T(x) = Ax$ is not one-to-one.

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

d) **T** **F** If A is an invertible $n \times n$ matrix, then the columns of A form a basis for $\mathbf{R}^n$.

IMT

e) **T** **F** Suppose A is a 4×5 matrix and the column span of A has dimension 2. Then the set of solutions to $Ax = 0$ is a 3-dimensional subspace of $\mathbf{R}^5$.

$\text{rank}(A) + \text{nullity}(A) = 5$

$2 + \text{nullity}(A) = 5$

$\text{nullity}(A) = 3$ ✓

Nul(A) lives in $\mathbb{R}^5$ ✓

Problem 2. [11 points]

All parts are unrelated. You do not need to show your work except in (d) and (e).

a) Complete the following definition (be mathematically precise!):
A transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ is *one-to-one* if...

for each $\vec{b}$ in $\mathbb{R}^m$, the equation $T(\vec{x}) = \vec{b}$ has at most one solution.

b) Which of the following are subspaces of $\mathbf{R}^3$? Circle all that apply.

(i) $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x - y - z = 2 \right\}$.

(ii) The set $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$. Not closed under addition or scalar multiplication $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ Not in the set.

(iii) The set of solutions to the equation $\begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 2 \end{pmatrix} v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

This is $\mathrm{Nul} \begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 2 \end{pmatrix}$.

c) Suppose A is a 5×7 matrix which is not the zero matrix, and suppose every column of A is the same. Fill in the blanks:

$\dim(\mathrm{Col}\, A) = 1$ rank$(A) =$ 1 nullity$(A) =$ 6 $(7-1)$

d) Suppose A is an invertible matrix and $A^{-1} = \begin{pmatrix} 1 & 5 \\ 2 & -1 \end{pmatrix}$.

Find all solutions (if there are any) to the equation $Ax = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

$\vec{x} = A^{-1} \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ $\vec{x} = \begin{pmatrix} 1 & 5 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -13 \\ 7 \end{pmatrix}$

e) Let $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v = \begin{pmatrix} 3 & -2 \end{pmatrix}$. Compute the product uv.

$uv = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 3 & -2 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 6 & -4 \end{pmatrix}$

2×1 1×2 2×2

Problem 3. [10 points]

You do not need to show your work on (c), (d), or (e).

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that reflects across the line $y = x$, and let $U : \mathbf{R}^3 \to \mathbf{R}^2$ be the transformation given by $U(x, y, z) = (x - 2y + z,\ 2x - 3z)$.

a) Find the standard matrix A for T.

$$A = \begin{pmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{pmatrix} \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

b) Find the standard matrix B for U.

$$B = \begin{pmatrix} U(\vec{e}_1) & U(\vec{e}_2) & U(\vec{e}_3) \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & 1 \\ 2 & 0 & -3 \end{pmatrix}$$

c) Is T invertible? YES NO

d) Is U onto? YES NO

e) Which of the following compositions makes sense?

$T \circ U$ $\qquad$ $U \circ T$.

f) Find the standard matrix for the composition you circled in (e).

$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 \\ 2 & 0 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix}$$

Problem 4.

[9 points]

Consider the following matrix A and its reduced row echelon form:

$$A = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 4 & 3 \\ 2 & -4 & -1 \\ 4 & -8 & -2 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} \boxed{1} & -2 & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let T be the matrix transformation $T(x) = Ax$.

a) Write a basis for the range of T.
(no work necessary on this part, and no partial credit)

Basis for Col A: $\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -1 \\ -2 \end{pmatrix} \right\}$

b) Find a basis for Nul A.

$x_1 - 2x_2 = 0 \qquad x_2 = x_2 \qquad x_3 = 0$

free

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ x_2 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

c) Are there nonzero vectors v and w, with $v \neq w$, so that $T(v) = T(w)$? If yes, write such vectors v and w. If no, justify why there are no such vectors v and w.

Yes. $T\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = T\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

d) Find one vector x so that $T(x) = \begin{pmatrix} 0 \\ 6 \\ -2 \\ -4 \end{pmatrix}$.

This is 2·(last column of A)

$$T\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ -2 \\ -4 \end{pmatrix}$$

Problem 5.

[10 points]

Parts (a), (b), and (c) are unrelated.

a) Suppose A is an $m \times n$ matrix and T is its associated matrix transformation $T(x) = Ax$. Which of the following are true? Circle all that apply.

(i) If T is one-to-one, then $\dim(\text{range}(T)) = n$. ✓ A has n pivots

(ii) If T is onto, then the equation $Ax = b$ is consistent for each b in $\mathbf{R}^m$.

Def. of onto

(iii) If $\{Ae_1, Ae_2, \ldots, Ae_n\}$ is linearly independent, then T is one-to-one.

This is just statement that columns of A are lin. independent

(iv) If T is one-to-one and onto, then $m = n$.

b) Write a matrix A so that the transformation $T(v) = Av$ has domain $\mathbf{R}^3$ and has range equal to the line $y = -2x$ in $\mathbf{R}^2$.

A 2×3

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}$$ for example

c) Write a 2×2 matrix A that satisfies $A^2 = -I_2$.

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$ or $$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Countercl. rotation 90°

clockwise rotation 90°

A^2 is rotation 180°

$A^2 = -I_2$.