# MATH 1553, SPRING 2020
# MIDTERM 2, LECTURE A1-A3 (8:00 - 8:50 AM)

| Name | Key A | Section | |
|---|---|---|---|

Please **read all instructions** carefully before beginning.

- Write your name on the front of each page (not just the cover page!).
- The maximum score on this exam is 50 points, and you have 50 minutes to complete this exam.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- As always, $e_1, e_2, \ldots, e_n$ refer to the standard unit coordinate vectors of $\mathbf{R}^n$.
- Show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit! If you cannot fit your work on the front side of the page, use the back side of the page and indicate that you are using the back side.
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All work must be written on the exam itself.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

## Problem 1.

[10 points]

True or false. Circle **T** if the statement is *always* true.
Otherwise, circle **F**. You do not need to show work or justify your answer.

a) **T** **F** If $\{v_1, v_2, v_3, v_4\}$ is a linearly independent set of vectors in $\mathbf{R}^4$, then $\{v_1, v_2, v_3, v_4\}$ must be a basis for $\mathbf{R}^4$.

Basis Theorem

b) **T** **F** If $A$ is a $4 \times 5$ matrix and the solution set to $Ax = 0$ is a line, then the matrix transformation $T(x) = Ax$ is onto.

$\text{rank}(A) + \text{nullity}(A) = 5$

$\text{rank}(A) + 1 = 5$

$\text{rank}(A) = 4$.

c) **T** **F** Suppose $A$ is a $3 \times 3$ matrix and the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is not in Col$(A)$.

Then the transformation $T(x) = Ax$ cannot be one-to-one.

T not onto and A $3 \times 3$

$\Rightarrow$ T not one-to-one, by IMT

d) **T** **F** If $A$ is a $3 \times 3$ matrix and $Ax = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ has exactly one solution, then every vector in $\mathbf{R}^3$ is in the span of the columns of $A$.

$A\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ has unique solution

$\Rightarrow A\vec{x} = \vec{0}$ has unique sol.

$\Rightarrow$ A invertible by IMT

$\Rightarrow \text{Col } A = \mathbb{R}^3$

e) **T** **F** If $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation and $n < m$, then the equation $T(x) = 0$ must have infinitely many solutions.

$T: \mathbb{R}^2 \to \mathbb{R}^3$

$T(x, y) = (x, y, 0)$

$T(\vec{x}) = \vec{0}$ has only the trivial solution.

## Problem 2.

[11 points]

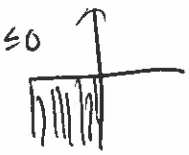
All parts are unrelated. You do not need to show your work except in (d) and (e).

**a)** Complete the following definition (be mathematically precise!):
A transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ is *one-to-one* if...

for each $\vec{b}$ in $\mathbb{R}^m$, the equation $T(\vec{x}) = \vec{b}$ has at most one solution

**b)** Let $W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; x \le 0 \text{ and } y \le 0 \right\}$. Which subspace properties does $W$ satisfy? Circle all that apply.

(i) $W$ contains the zero vector. ✓ $0 \le 0$ and $0 \le 0$



(ii) $W$ is closed under addition. ✓

(iii) $W$ is closed under scalar multiplication. $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ in $W$ but $-1\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not in $W$.

**c)** Suppose $A$ is an $8 \times 6$ matrix which is not the zero matrix, and suppose every column of $A$ is the same. Fill in the blanks:

So rank$(A) = 1$

rank$(A) =$ 1 nullity$(A) =$ 5. $(= 6 - 1)$

**d)** Suppose $A$ is an invertible matrix and $A^{-1} = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}$.

Find all solutions (if there are any) to the equation $Ax = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

$$\vec{x} = A^{-1}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 22 \end{pmatrix}$$

**e)** Suppose $A$ and $B$ are $n \times n$ matrices. Which of the following statements are true? Circle all that apply.

(i) If $A$ and $B$ are invertible, then $(AB)^{-1} = B^{-1}A^{-1}$.

(ii) If $A$ is invertible and $AB = 0$, then $B = 0$.

$$AB = 0$$

So $\cancel{A^{-1}}(\cancel{A}B) = A^{-1}(0)$

$$I_n B = 0$$

$$B = 0.$$

## Problem 3. [10 points]

You do not need to show your work on (c), (d), or (e).

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that reflects across the line $y = x$, and let $U : \mathbf{R}^2 \to \mathbf{R}^3$ be the transformation given by $U(x, y) = (x - 2y,\ y - 2x,\ 3y)$.

**a)** Find the standard matrix $A$ for $T$.

$$A = \left( T\begin{pmatrix}1\\0\end{pmatrix} \ T\begin{pmatrix}0\\1\end{pmatrix} \right) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

**b)** Find the standard matrix $B$ for $U$.

$$B = \left( U\begin{pmatrix}1\\0\end{pmatrix} \ U\begin{pmatrix}0\\1\end{pmatrix} \right) = \begin{pmatrix}1 & -2\\ -2 & 1\\ 0 & 3\end{pmatrix}$$

**c)** Is $T$ invertible? YES NO

**d)** Is $U$ one-to-one? YES NO

**e)** Which of the following compositions makes sense?

$T \circ U$ $\quad$ $U \circ T.$

**f)** Find the standard matrix for the composition you circled in (e).

$$BA = \begin{pmatrix}1 & -2\\ -2 & 1\\ 0 & 3\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$= \begin{pmatrix}-2 & 1\\ 1 & -2\\ 3 & 0\end{pmatrix}$$

## Problem 4.

[9 points]

Consider the following matrix $A$ and its reduced row echelon form:

$$A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ -3 & -6 & 3 & 3 \\ 2 & 4 & -2 & -2 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $T$ be the matrix transformation $T(x) = Ax$.

**a)** Write a basis for the range of $T$.
(no work necessary on this part, and no partial credit)

A Basis for Col (A) is $\left\{ \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} \right\}$

**b)** Find a basis for Nul $A$.

$$(A | \vec{0}) \rightsquigarrow \left(\begin{array}{cccc|c} \boxed{1} & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 + 2x_2 - x_3 = 0$

$x_2 = x_2 \quad x_3 = x_3$

free

$x_4 = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2x_2 + x_3 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Basis: $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

**c)** Are there nonzero vectors $v$ and $w$, with $v \neq w$, so that $T(v) = T(w)$? If yes, write such vectors $v$ and $w$. If no, justify why there are no such vectors $v$ and $w$.

Yes. $T\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} = T\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

for example

**d)** Find one vector $x$ that satisfies $T(x) = e_1$.

Note (1st column of A) + (4th Column of A) = $\vec{e}_1$

So $T\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \vec{e}_1$. More answers possible

## Problem 5.

[10 points]

Parts (a), (b), and (c) are unrelated.

**a)** Suppose $A$ is an $m \times n$ matrix and $T$ is its associated matrix transformation $T(x) = Ax$. Which of the following are true? Circle all that apply.

(i) If $T$ is one-to-one, then for each $b$ in $\mathbf{R}^m$, the equation $Ax = b$ has exactly one solution.

(ii) If $T$ is onto, then the dimension of the range of $T$ must equal $n$. ← must equal $m$.

(iii) If $\{Ae_1, Ae_2, \ldots, Ae_n\}$ is linearly dependent, then $T$ is not onto. ← $T(\vec{x}) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{x}$ is onto

(iv) If $T$ is one-to-one and onto, then $m = n$.

**b)** Write a single matrix $A$ that satisfies both of the following properties:

- $\dim(\mathrm{Nul}\ A) = 3$.
- The range of the transformation $T(v) = Av$ is the line $y = 2x$ in $\mathbf{R}^2$.

rank + nullity is $3+1=4$, so $A$ is $2\times4$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}$$ for example.

**c)** Write a $2 \times 2$ matrix $A$ so that $A^2 = -I_2$.

For example, $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ counterCl. rotation by $90°$

or $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ clock. rot. by $90°$

Either way, $A^2$ is rotation by $180°$

$A^2 = -I_2$.