MATH 1553, JANKOWSKI, 11:15 AM (C1-C4)
MIDTERM 1, SPRING 2020

Name	Key	Section	

Please **read all instructions** carefully before beginning.

- Write your name on the front of each page (not just the cover page!).
- The maximum score on this exam is 50 points, and you have 50 minutes to complete this exam.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit! If you cannot fit your work on the front side of the page, use the back side of the page and indicate that you are using the back side.
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All work must be written on the exam itself.
- You may cite any theorem proved in class or in the sections we covered the text.
- Good luck!

Problem 1.

[10 points]

These problems are true or false. Circle **T** if the statement is *always* true. Otherwise, circle **F**. You do not need to show work or to justify your answer.

a) **T** **F** The augmented matrix below is in RREF.

$$\left(\begin{array}{ccc|c} 1 & -3 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

b) **T** **F** If b is a vector in $\mathbf{R}^3$, then b must be a linear combination of the vectors $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} \boxed{1} & 1 & 0 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

These Vectors don't span $\mathbb{R}^3$

c) **T** **F** Suppose A is a 3×3 matrix and b is in $\mathbf{R}^3$. If the solution set to $Ax = b$ is Span $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$, then b is the zero vector.

Span$\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$ includes $0\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \vec{0}$.

So $A\vec{0} = \vec{b}$ then $\vec{b} = \vec{0}$.

d) **T** **F** Suppose A is an $m \times n$ matrix and the equation $Ax = 0$ has only the trivial solution. Then A must have exactly m pivots.

$\underline{n}$ pivots (pivot in each column)

e) **T** **F** If v_1, v_2, v_3 are vectors in $\mathbf{R}^3$ and the vector equation

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = b$$

is inconsistent for some b in $\mathbf{R}^3$, then $\{v_1, v_2, v_3\}$ is linearly dependent.

three vectors in $\mathbb{R}^3$ are lin. dependent if they don't span $\mathbb{R}^3$

Problem 2. [11 points]

Short answer. You do not need to show your work except in part (a).

a) (2 points) Let $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \\ 0 & 2 \end{pmatrix}$ and $x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Compute Ax.

$$1\begin{pmatrix}3\\-1\\0\end{pmatrix} - 1\begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}3\\-1\\0\end{pmatrix} + \begin{pmatrix}-1\\-1\\-2\end{pmatrix} = \begin{pmatrix}2\\-2\\-2\end{pmatrix}$$

b) (2 points) Complete the following definition (be mathematically precise!): Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbf{R}^n$. The *span* of $v_1, v_2, \ldots, v_p$, which we denote by $\mathrm{Span}\{v_1, \ldots, v_p\}$, is...

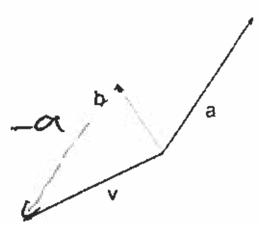
$$\mathrm{Span}\{\vec{v}_1, \ldots, \vec{v}_p\} = \{x_1\vec{v}_1 + \cdots + x_p\vec{v}_p \mid x_1, \ldots, x_p \text{ real}\}.$$

c) (2 points) Suppose v_1 and v_2 are vectors in $\mathbf{R}^3$. Which of the following statements must be true? Circle all that apply.

(i) If the set $\{v_1, v_2\}$ of vectors in $\mathbf{R}^3$ is linearly independent, then $\mathrm{Span}\{v_1, v_2\} = \mathbf{R}^2$. Span is a plane in $\mathbb{R}^3$, NOT $\mathbb{R}^2$

(ii) There must be at least one vector b in $\mathbf{R}^3$ which is not a linear combination of v_1 and v_2.

d) (2 points) Consider the vectors a, b, and v below.



Which of the following describes v in terms of a and b?

(i) $v = a + b$ (ii) $v = a - b$ (iii) $v = b - a$ (iv) $v = -a - b$

e) [3 points] Suppose we are given a consistent system of 3 linear equations in 4 variables. Which of the following are *possible* for the solution set of the system? Circle all that apply.

$A\vec{x} = \vec{b}$

A 3×4

Mtx A Can't have pivot in each column

(i) The system has a unique solution.

(ii) The solution set is a line in $\mathbf{R}^4$.

Sol. set lives in $\mathbb{R}^4$

(iii) The solution set is a plane in $\mathbf{R}^3$.

Sol. set doesn't live in $\mathbb{R}^3$

Problem 3.

[9 points]

Show your work in part (a).

a) (4 points)

Is $\begin{pmatrix}-1\\-1\\-3\end{pmatrix}$ in the span of $\begin{pmatrix}1\\3\\-1\end{pmatrix}$ and $\begin{pmatrix}1\\2\\1\end{pmatrix}$? If your answer is yes, write $\begin{pmatrix}-1\\-1\\-3\end{pmatrix}$ as a linear combination of $\begin{pmatrix}1\\3\\-1\end{pmatrix}$ and $\begin{pmatrix}1\\2\\1\end{pmatrix}$. If your answer is no, justify why.

$\vec{v}_1$ $\vec{v}_2$ $\vec{b}$

$x_1\vec{v}_1 + x_2\vec{v}_2 = \vec{b}$

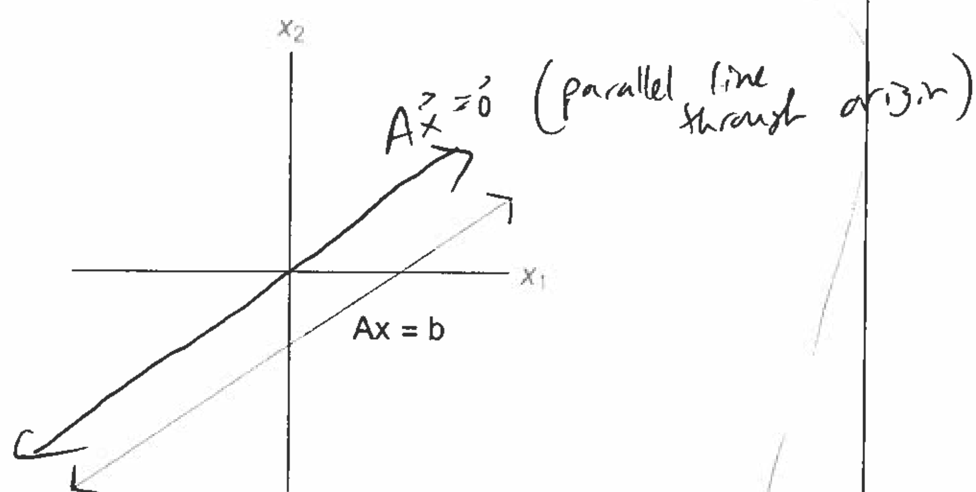
$$\left(\begin{array}{cc|c} 1 & 1 & -1 \\ 3 & 2 & -1 \\ -1 & 1 & -3 \end{array}\right) \xrightarrow[R_3 = R_3 + R_1]{R_2 = R_2 - 3R_1} \left(\begin{array}{cc|c} \boxed{1} & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 2 & -4 \end{array}\right) \xrightarrow[R_2 = -R_2]{\text{Destroy } R_3} \left(\begin{array}{cc|c} \boxed{1} & 1 & -1 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 = R_1 - R_2} \left(\begin{array}{cc|c} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & 0 \end{array}\right)$$

Yes! $x_1 = 1$ $x_2 = -2$

$$\boxed{1\begin{pmatrix}1\\3\\-1\end{pmatrix} - 2\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\\-3\end{pmatrix}}$$

b) (5 points) Suppose A is a 3×2 matrix and the solution set to $Ax = b$ is drawn below for some vector b.



(i) Draw the solution set to $Ax = 0$ on the same graph above.

(ii) Fill in the two blanks below:

Geometrically, the span of the columns of A is a line in $\mathbb{R}^{\boxed{3}}$.

(one pivot column, one non-pivot column by part (a))

Problem 4.

[11 points]

Part (a) is free response. Show your work or you may receive little or no credit, even if your answer is correct.

a) Brian O'Blivion has given you the following system of equations:

$$\begin{aligned} x_1+3x_2-x_3-x_4 &= -1 \\ -2x_1-6x_2+3x_3+x_4 &= 3 \\ 2x_1+6x_2-x_3-3x_4 &= -1. \end{aligned}$$

(i) (5 points) Write the system as an augmented matrix, and put the matrix into reduced row echelon form.

$$\left(\begin{array}{cccc|c} 1 & 3 & -1 & -1 & -1 \\ -2 & -6 & 3 & 1 & 3 \\ 2 & 6 & -1 & -3 & -1 \end{array}\right) \xrightarrow[R_3=R_3-2R_1]{R_2=R_2+2R_1} \left(\begin{array}{cccc|c} \boxed{1} & 3 & -1 & -1 & -1 \\ 0 & 0 & \boxed{1} & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow[R_1=R_1+R_2]{R_3=R_3-R_2} \left(\begin{array}{cccc|c} \boxed{1} & 3 & 0 & -2 & 0 \\ 0 & 0 & \boxed{1} & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

(ii) (4 points) Write the solution set to the system of equations in parametric vector form. Clearly indicate which variables (if any) are free.

$x_1+3x_2-2x_4=0$ $\quad x_2=x_2$ free $\quad x_3=1+x_4 \quad x_4=x_4$ (free)

$x_1=-3x_2+2x_4$

$$\begin{pmatrix} x_1\\x_2\\x_3\\x_4 \end{pmatrix} = \begin{pmatrix} -3x_2+2x_4\\x_2\\1+x_4\\x_4 \end{pmatrix} = \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix} + x_2\begin{pmatrix} -3\\1\\0\\0 \end{pmatrix} + x_4\begin{pmatrix} 2\\0\\1\\1 \end{pmatrix}$$

b) (Unrelated to (a), worth 2 points) Suppose A is a matrix and b is a vector so that the solutions to the equation $Ax=b$ have parametric form

$$x_1=-1-2x_3, \qquad x_2=2+5x_3, \qquad x_3=x_3\ (x_3 \text{ real}).$$

Write one nonzero vector x that satisfies $Ax=0$. You don't need to show your work for this part.

Hom. sol. is Span $\left\{\begin{pmatrix} -2\\5\\1 \end{pmatrix}\right\}$

$\vec{x}=\begin{pmatrix} -2\\5\\1 \end{pmatrix}$ or any nonzero scalar multiple of $\begin{pmatrix} -2\\5\\1 \end{pmatrix}$

Problem 5. [9 points]

Parts (a) and (b) are unrelated. Show your work on part (a).

a) (5 points) Find all real values of h (if there are any) so that the following set of vectors $\{v_1, v_2, v_3\}$ is linearly independent:

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} -1 \\ h \\ h \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & h \\ 3 & 5 & h \end{pmatrix} \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & h+2 \\ 0 & 5 & h+3 \end{pmatrix}$$

$$\xrightarrow{R_3 = R_3 - 5R_2} \begin{pmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & h+2 \\ 0 & 0 & h+3-5(h+2) \end{pmatrix}$$

lin. independence $\Rightarrow$ $h+3-5(h+2) \neq 0$.

$h+3-5(h+2) = -4h-7$

$-4h-7 \neq 0$

$4h \neq -7$

all real h so that $\boxed{h \neq -\frac{7}{4}}$

b) (4 points) Write a matrix A so that the solution set to $Ax = 0$ is a line in $\mathbf{R}^3$ and the equation $Ax = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$ is consistent.

Many examples possible, like.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A must:

1. Be 4×3
2. Have exactly one free variable for $A\vec{x} = \vec{0}$ (thus, 2 pivot columns)
3. Have $\begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$ in its column span.