MATH 1553, JANKOWSKI, 8:00 AM (A1-A3)
MIDTERM 1, SPRING 2020

Name	Key	Section	

Please **read all instructions** carefully before beginning.

- Write your name on the front of each page (not just the cover page!).
- The maximum score on this exam is 50 points, and you have 50 minutes to complete this exam.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit! If you cannot fit your work on the front side of the page, use the back side of the page and indicate that you are using the back side.
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All work must be written on the exam itself.
- You may cite any theorem proved in class or in the sections we covered the text.
- Good luck!

Problem 1.

[10 points]

These problems are true or false. Circle **T** if the statement is *always* true. Otherwise, circle **F**. You do not need to show work or to justify your answer.

a) **T** **F** The augmented matrix below is in RREF.

$$\left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

b) **T** **F** Suppose $w_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $w_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, and $w_3 = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}$.

Then $\mathrm{Span}\{w_1, w_2, w_3\}$ is $\mathbf{R}^3$.

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 2 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} \boxed{1} & 0 & -2 \\ 0 & \boxed{1} & 3 \\ 0 & 0 & \boxed{-1} \end{pmatrix}$$

Pivot in every rows, so vectors span $\mathbb{R}^3$.

c) **T** **F** If v_1, v_2, v_3 are vectors in $\mathbf{R}^3$ and the vector equation

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = b$$

is inconsistent for some b in $\mathbf{R}^3$, then $\{v_1, v_2, v_3\}$ is linearly dependent.

$\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathbb{R}^3$ if $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ lin. ind. (by increasing span criterion).

d) **T** **F** Suppose A is a 2×3 matrix and b is a vector so that $Ax = b$ is consistent. Then $-4b$ must be a linear combination of the columns of A.

$\vec{b}$ is in column span of A, thus $-4\vec{b}$ is also in col. span of A.

e) **T** **F** If A is an $m \times n$ matrix and $n > m$, then the equation $Ax = b$ must be inconsistent for some b in $\mathbf{R}^m$.

$m = 2$ $n = 3$

$$A = \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \end{pmatrix}$$

Columns of A span $\mathbb{R}^2$.

Problem 2.

[11 points]

Short answer. You do not need to show your work except in part (a).

a) (2 points) Let $A = \begin{pmatrix} -2 & 1 \\ 2 & 1 \\ 4 & 0 \end{pmatrix}$ and $x = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Compute Ax.

$$A\vec{x} = -1\begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix}$$

b) (2 points) Complete the following definition (be mathematically precise!): Let $w, v_1, v_2, \ldots, v_p$ be vectors in $\mathbf{R}^n$. We say w is a *linear combination* of $v_1, v_2, \ldots, v_p$ if...

$\vec{w} = x_1\vec{v}_1 + \cdots + x_p\vec{v}_p$ for some scalars $x_1, \ldots, x_p$.

c) (2 points) Suppose v_1, v_2, v_3 are vectors in $\mathbf{R}^4$. Which of the following statements must be true? Circle all that apply.

(i) If $\{v_1, v_2, v_3\}$ is linearly dependent, then the vector equation

$$x_1v_1 + x_2v_2 + x_3v_3 = 0$$

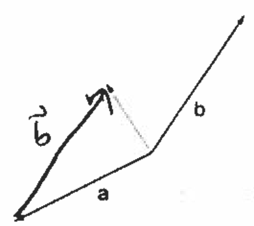
has a solution satisfying $x_1 \neq 0$, $x_2 \neq 0$, and $x_3 \neq 0$.

$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\vec{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$

$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$ requires $\underline{x_1 = 0}$.

(ii) If $\{v_1, v_2, v_3\}$ is linearly independent, then $\{v_1, v_2\}$ is linearly independent.

d) (2 points) Consider the vectors a, b, and v below.



$\vec{a} + \vec{b} = \vec{v}$

Which of the following describes v in terms of a and b?

(i) $v = a + b$ (ii) $v = a - b$ (iii) $v = b - a$ (iv) $v = -a - b$

e) (3 points) Suppose we are given a consistent system of 4 linear equations in 3 variables. Which of the following are *possible* for the solution set of the system? Circle all that apply.

(i) The system has a unique solution.

(ii) The solution set forms a line in $\mathbf{R}^4$.

(iii) The solution set forms a plane in $\mathbf{R}^3$.

Solution is $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. lives in $\mathbb{R}^3$

Can be unique solution, e.g.

$$\left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Problem 3.

[9 points]

Show your work in part (a).

$x_1 \vec{v_1} + x_2 \vec{v_2} = \vec{b}$

a) (4 points)

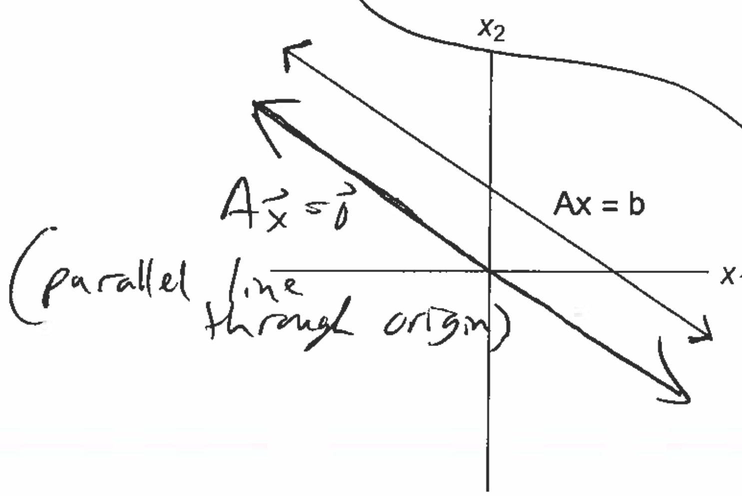
Is $\begin{pmatrix} 1 \\ 5 \\ -11 \end{pmatrix}$ in the span of $\begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$? If your answer is yes, write $\begin{pmatrix} 1 \\ 5 \\ -11 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$. If your answer is no, justify why.

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 1 & 5 \\ -5 & -4 & -11 \end{array}\right) \xrightarrow[R_3 = R_3 + 5R_1]{R_2 = R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -3 & 3 \\ 0 & 6 & -6 \end{array}\right) \xrightarrow[\text{Then } R_2 = \frac{R_2}{-3}]{\text{Destroy } R_3} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right) \qquad x_1 = 3 \qquad x_2 = -1$$

$$3\begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ -11 \end{pmatrix}$$

b) (5 points) Suppose A is a 4 × 2 matrix and the solution set to $Ax = b$ is drawn below for some vector b.



(i) Draw the solution set to $Ax = 0$ on the same graph above.

Column span lives in $\mathbb{R}^4$

(ii) Fill the blanks below:

Geometrically, the span of the columns of A is a line in $\mathbb{R}^{\boxed{4}}$.

(A has 1 pivot column and $\underbrace{1 \text{ non-pivot column}}$, from (a))

[$A\vec{x} = \vec{b}$ had 1 free variable]

Problem 4. [11 points]

Part (a) is free response. Show your work or you may receive little or no credit, even if your answer is correct.

a) Snidely Whiplash has given you the following consistent system of equations:

$$\begin{aligned} x_1 - x_2 - 4x_3 - x_4 &= 4 \\ -x_1 + 2x_2 + 8x_3 + 3x_4 &= -5 \\ 2x_1 - x_2 - 4x_3 &= 7. \end{aligned}$$

(i) (5 points) Write the system as an augmented matrix, and put the matrix into reduced row echelon form.

$$\left(\begin{array}{cccc|c} 1 & -1 & -4 & -1 & 4 \\ -1 & 2 & 8 & 3 & -5 \\ 2 & -1 & -4 & 0 & 7 \end{array}\right) \xrightarrow[R_3 = R_3 - 2R_1]{R_2 = R_2 + R_1} \left(\begin{array}{cccc|c} \boxed{1} & -1 & -4 & -1 & 4 \\ 0 & 1 & 4 & 2 & -1 \\ 0 & 1 & 4 & 2 & -1 \end{array}\right)$$

$$\xrightarrow[R_1 = R_1 + R_2]{R_3 = R_3 - R_2} \left(\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 1 & 3 \\ 0 & \boxed{1} & 4 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

(ii) (4 points) Write the solution set to the system of equations in parametric vector form. Clearly indicate which variables (if any) are free.

$x_1 = 3 - x_4 \qquad x_2 = -1 - 4x_3 - 2x_4 \qquad x_3 = x_3 \quad x_4 = x_4$ (free)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 - x_4 \\ -1 - 4x_3 - 2x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

b) (Unrelated to (a), worth ~~2 points)~~ Is there a 2×2 matrix A so that the solution set to $Ax = 0$ is the line $x_1 - x_2 = 3$? If your answer is yes, write such a matrix A. If your answer is no, justify why there is no such matrix A.

No. $\vec{x} = \vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ must be solution to $A\vec{x} = \vec{0}$,

but $0 - 0 \neq 3$. (Line doesn't pass through origin)

Problem 5. [9 points]

Parts (a) and (b) are unrelated. Show your work in part (a).

a) (5 points) Let v_1, v_2, v_3 be the vectors below.

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -2 \\ 8 \\ 6 \end{pmatrix}.$$

Show that $\{v_1, v_2, v_3\}$ is linearly dependent, and find a linear dependence relation for the vectors v_1, v_2, v_3.

$$\left(\begin{array}{ccc|c} 0 & 1 & -2 & 0 \\ 1 & -2 & 8 & 0 \\ 2 & 1 & 6 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & -2 & 8 & 0 \\ 0 & 1 & -2 & 0 \\ 2 & 1 & 6 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 - 2R_1} \left(\begin{array}{ccc|c} 1 & -2 & 8 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 5 & -10 & 0 \end{array}\right) \xrightarrow{R_3 = R_3 - 5R_2} \left(\begin{array}{ccc|c} \boxed{1} & -2 & 8 & 0 \\ 0 & \boxed{1} & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\boxed{\text{Inf. many sol. to } x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}}$

$$\xrightarrow{R_1 = R_1 + 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 = -4x_3$, $x_2 = 2x_3$, $x_3 = x_3$ (free)

when $x_3 = 1$: get

$$\boxed{-4\vec{v}_1 + 2\vec{v}_2 + \vec{v}_3 = \vec{0}.}$$

b) (4 points) Write a matrix A so that the solution set to $Ax = 0$ is a line in $\mathbf{R}^3$ and the equation $Ax = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ is consistent.

Many examples possible, for example

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

① A must be 4×3,

② have two pivots (one free var. for $A\vec{x} = \vec{0}$)

③ Have $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ in its column span.