MATH 1553, SPRING 2020
SAMPLE MIDTERM 3B: COVERS 4.1 THROUGH 5.6

Name		Section	

Please **read all instructions** carefully before beginning.

- Our actual midterm 3 will take place on Canvas, and you will have 75 minutes to complete it. Before the end of the day on Friday, April 10, Sample Midterm 3A will be posted on Canvas.
- This practice exam (3B) is meant to be completed in 50 minutes or less. It is meant as additional practice for our midterm.
- As always, RREF means "reduced row echelon form".

The exam is not designed to test material from the previous midterm on its own. However, knowledge of the material prior to section §4.1 is necessary for everything we do for the rest of the semester, so it is fair game for the exam as it applies to §§4.1 through 5.6.

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Problem 1.

Answer true if the statement is *always* true. Otherwise, answer false. In every case, assume that the entries of the matrix A are real numbers.

a) **T** **F** If A is the 3×3 matrix satisfying $Ae_1 = e_2$, $Ae_2 = e_3$, and $Ae_3 = e_1$, then $\det(A) = 1$.

b) **T** **F** If A is an $n \times n$ matrix and $\det(A) = 2$, then 2 is an eigenvalue of A.

c) **T** **F** If A and B are $n \times n$ matrices with $\det(A) = 0$ and $\det(B) = 0$, then $\det(A+B) = 0$.

d) **T** **F** If A is an $n \times n$ matrix and v and w are eigenvectors of A, then $v + w$ is also an eigenvector of A.

e) **T** **F** It is possible for a lower-triangular matrix A to have a non-real complex eigenvalue.

Solution.

a) True. $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. You can compute $\det(A) = 1$ or just do two row swaps to get the identity matrix, so that $\det(A) = (-1)^2 = 1$.

b) False. For example, $A = \begin{pmatrix} 4 & 0 \\ 0 & 1/2 \end{pmatrix}$ has $\det(A) = 2$ but its eigenvalues are 4 and $\frac{1}{2}$.

c) False. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

d) False. For example, if $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ then $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are eigenvectors, but $A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ so $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not an eigenvector.

e) False. Since A is a lower-triangular matrix, its diagonal entries (which are real numbers) are its eigenvalues.

Extra space for scratch work on problem 1

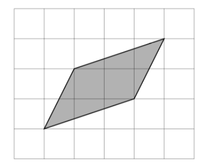
Problem 2.

Short answer. Show your work on part (c). In every case, the entries of each matrix must be real numbers.

a) Write a 2×2 matrix A which is invertible but not diagonalizable.

b) Write a 2×2 matrix A for which $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are eigenvectors corresponding to the same eigenvalue.

c) Find the area of the parallelogram drawn below (the grid marks are spaced one unit apart).



d) Write a 3×3 matrix A with only one real eigenvalue $\lambda = 4$, such that the 4-eigenspace for A is a two-dimensional plane in $\mathbf{R}^3$.

Solution.

a) Many answers possible. For example, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

b) Any scalar multiple of the identity will work, for example $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.

c) The area is $\det \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = 5$.

d) Many examples possible. For example, $A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}$.

Extra space for work on problem 2

Problem 3.

Parts (a) and (b) are unrelated.

a) Consider the matrix

$$A = \begin{pmatrix} 3 & -7 \\ 1 & -1 \end{pmatrix}$$

Find all eigenvalues of A. Simplify your answer. For the eigenvalue with negative imaginary part, find an eigenvector.

b) Let $A = \begin{pmatrix} 7 & -8 \\ 4 & -5 \end{pmatrix}$. Find a formula for A^n and simplify your answer completely.

Solution.

a) We compute the characteristic equation:

$$0 = \det(A - \lambda I) = (3-\lambda)(-1-\lambda) + 7 = \lambda^2 - 2\lambda - 3 + 7 = \lambda^2 - 2\lambda + 4.$$

By the quadratic formula,

$$\lambda = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm 2i\sqrt{3}}{2} = 1 \pm \sqrt{3}i.$$

Let $\lambda = 1 - \sqrt{3}i$. Then

$$\left(A - \lambda I \mid 0 \right) = \left(\begin{array}{cc|c} 2+\sqrt{3}i & -7 & 0 \\ \star & \star & 0 \end{array} \right) \xrightarrow[R_1 = R_1/(2+\sqrt{3}i)]{\text{destroy } R_2} \left(\begin{array}{cc|c} 1 & \frac{-7}{2+\sqrt{3}i} & 0 \\ 0 & 0 & 0 \end{array} \right) = \left(\begin{array}{cc|c} 1 & -2+\sqrt{3}i & 0 \\ 0 & 0 & 0 \end{array} \right)$$

So $x_1 = (2 - \sqrt{3}i)x_2$ and x_2 is free. An eigenvector is $v = \begin{pmatrix} 2-\sqrt{3}i \\ 1 \end{pmatrix}$.

An alternative method for finding an eigenvector, using a trick you may have seen in class, is to take the first row $(a \ b)$ of $A - \lambda I_2$ to get an eigenvector $\begin{pmatrix} -b \\ a \end{pmatrix}$:

$$A - \lambda I_2 = \begin{pmatrix} 2+\sqrt{3}i & -7 \\ \star & \star \end{pmatrix}$$

Thus $\begin{pmatrix} 7 \\ 2+\sqrt{3}i \end{pmatrix}$ is an eigenvector for λ. This answer is equivalent to our answer v above since it is a nonzero scalar multiple of v, as $\begin{pmatrix} 7 \\ 2+\sqrt{3}i \end{pmatrix} = (2+\sqrt{3}i)v$.

b) The characteristic equation is $\lambda^2 - 2\lambda - 3 = 0$, and we find the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -1$. Row-reducing $(A - 3I \mid 0)$ and $(A + I \mid 0)$ gives $v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

as eigenvectors for λ_1 and λ_2, respectively. With $P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$, we get $A = PDP^{-1}$, so

$$\begin{aligned}
A^n &= PD^nP^{-1} \\
&= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3^n & 0 \\ 0 & (-1)^n \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3^n & -3^n \\ (-1)^{n+1} & 2(-1)^n \end{pmatrix} \\
&= \begin{pmatrix} 2 \cdot 3^n + (-1)^{n+1} & -2 \cdot 3^n + 2(-1)^n \\ 3^n + (-1)^{n+1} & -3^n + 2(-1)^n \end{pmatrix}.
\end{aligned}$$

Extra space for work on problem 3

Problem 4.

Let $A = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 2 & 0 \\ 3 & 0 & 4 \end{pmatrix}$.

a) Find the eigenvalues of A.

b) Find a basis for each eigenspace of A. Mark your answers clearly.

c) Is A diagonalizable? If your answer is yes, find a diagonal matrix D and an invertible matrix C so that $A = CDC^{-1}$. If your answer is no, justify why A is not diagonalizable.

Solution.

a) We solve $0 = \det(A - \lambda I)$.

$$0 = \det\begin{pmatrix} -1-\lambda & 0 & -2 \\ 0 & 2-\lambda & 0 \\ 3 & 0 & 4-\lambda \end{pmatrix} = (2-\lambda)(-1)^4 \det\begin{pmatrix} -1-\lambda & -2 \\ 3 & 4-\lambda \end{pmatrix}$$
$$= (2-\lambda)\left((-1-\lambda)(4-\lambda)+6\right) = (2-\lambda)(\lambda^2-3\lambda-4+6)$$
$$= (2-\lambda)(\lambda^2-3\lambda+2) = (2-\lambda)(\lambda-2)(\lambda-1)$$

So $\lambda = 1$ and $\lambda = 2$ are the eigenvalues.

$\underline{\lambda = 1}$: $\left(A - I \mid 0\right) = \left(\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 3 & 0 \end{array}\right) \xrightarrow[\text{then } R_1 = -R_1/2]{R_3 = R_3 + \frac{3}{2}R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ with solution $x_1 = -x_3$, $x_2 = 0$, $x_3 = x_3$. The 1-eigenspace has basis $\left\{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$.

$\underline{\lambda = 2}$:

$$\left(A - 2I \mid 0\right) = \left(\begin{array}{ccc|c} -3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 2 & 0 \end{array}\right) \xrightarrow[\text{then } R_1 = -R_1/3]{R_3 = R_3 + R_1} \left(\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

with solution $x_1 = -\frac{2}{3}x_3,\ x_2 = x_2,\ x_3 = x_3$. The 2-eigenspace has basis $\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2/3 \\ 0 \\ 1 \end{pmatrix}\right\}$.

b) A is diagonalizable; $A = CDC^{-1}$ where $C = \begin{pmatrix} -1 & 0 & -2/3 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

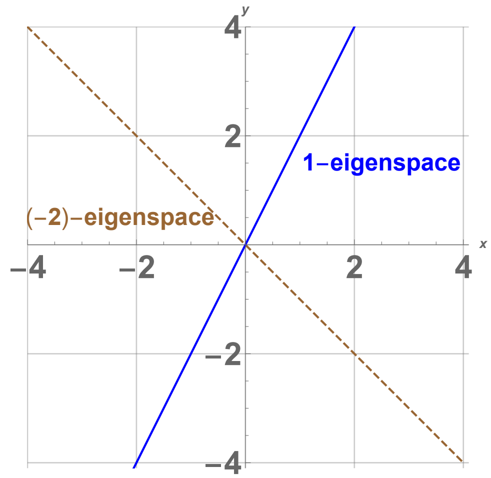
Extra space for work on problem 4

Problem 5.

Parts (a) and (b) are not related.

a) Find $\det(A^3)$ if $A = \begin{pmatrix} 1 & -3 & 4 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 1 & 2 & 3 \\ 2 & 0 & -1 & 20 \end{pmatrix}$.

b) Find the 2×2 matrix A whose eigenspaces are drawn below. Fully simplify your answer. (to be clear: the dashed line is the (-2)-eigenspace).



Solution.

a) Using the cofactor expansion along the second row, we find

$$\det(A) = -2(-1)^5 \det\begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & 3 \\ 2 & 0 & 20 \end{pmatrix} = 2\big(20 + 3(-6) + 2(-2)\big) = 2(20-18-4) = -4,$$

so $\det(A^3) = (-4)^3 = -64$.

b) From the picture, we see $\lambda_1 = 1$ has eigenvector $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Also, $\lambda = -2$ has eigenvector $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Forming $C = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$ we get $A = CDC^{-1}$, so

$$\begin{aligned} A &= \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} -3 & 3 \\ 6 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}. \end{aligned}$$

Extra space for work on problem 5