# MATH 1553, SPRING 2020
# SAMPLE MIDTERM 1B: COVERS THROUGH SECTION 2.5

| **Name** | | **Section** | |
|---|---|---|---|

Please **read all instructions** carefully before beginning.

- Write your name on the front of each page (not just the cover page!).
- The maximum score on this exam is 50 points, and you have 50 minutes to complete this exam.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means “reduced row echelon form."
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit! If you cannot fit your work on the front side of the page, use the back side of the page and indicate that you are doing so.
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All work must be written on the exam itself.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 50 minutes, without notes or distractions.

## Scoring Page

Please do not write on this page.

| 1 | 2 | 3 | 4 | 5 | Total |
|---|---|---|---|---|---|
| | | | | | |

## Problem 1.

[2 points each]

In parts (c) and (e), $A$ denotes an $m \times n$ matrix ($m$ rows and $n$ columns), and in (e), $b$ is a vector in $\mathbf{R}^m$. In (b)–(e), circle **T** if the statement is necessarily true, and circle **F** otherwise.

a) What is the best way to describe the solution set of the equation $x+2y-z=0$?

a line in $\mathbf{R}^2$ a line in $\mathbf{R}^3$ a plane in $\mathbf{R}^2$ a plane in $\mathbf{R}^3$

b) **T** **F** The following matrix has three pivots:

$$\left(\begin{array}{ccc|c} 1 & 7 & 2 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 15 \end{array}\right)$$

c) **T** **F** It is possible for the matrix equation $Ax = 0$ to be inconsistent.

d) **T** **F** The following matrix corresponds to a linear system with one free variable:

$$\left(\begin{array}{ccc|c} 1 & 7 & 2 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

e) **T** **F** The solution set of $Ax = b$ is empty or it is a translate of a span in $\mathbf{R}^m$.

## Solution.

**a)** The variables $y$ and $z$ are free, so the solution set is a plane. The total number of variables is 3, so it's a plane in $\mathbf{R}^3$.

**b)** **True.** The entries with 1 and 15 are all pivots.

**c)** **False.** The zero vector is a solution.

**d)** **True.** There is one (non-augmented) column without a pivot.

**e)** **False.** It is empty or it is a translate of a span in $\mathbf{R}^n$.

## Problem 2.

Consider the following system of linear equations:

$$\begin{aligned} 2x + \phantom{2}y + 12z &= \phantom{-}1 \\ x + 2y + \phantom{1}9z &= -1. \end{aligned}$$

**a)** [1 point ] Write the system as a vector equation.

**b)** [1 point ] Write the system as a matrix equation.

**c)** [1 point ] Write the system as an augmented matrix.

**d)** [4 points] Find the solution set in parametric vector form.

**e)** [3 points] Draw a picture of the solution set.

## Solution.

**a)** $x\begin{pmatrix}2\\1\end{pmatrix}+y\begin{pmatrix}1\\2\end{pmatrix}+z\begin{pmatrix}12\\9\end{pmatrix}=\begin{pmatrix}1\\-1\end{pmatrix}$

**b)** $\begin{pmatrix}2&1&12\\1&2&9\end{pmatrix}x=\begin{pmatrix}1\\-1\end{pmatrix}$

**c)** $\left(\begin{array}{ccc|c}2&1&12&1\\1&2&9&-1\end{array}\right)$

**d)** After row reduction, we obtain the augmented matrix

$$\left(\begin{array}{ccc|c}1&0&5&1\\0&1&2&-1\end{array}\right).$$

This translates into the equations

$$\begin{aligned} x \phantom{+ y} + 5z &= \phantom{-}1 \\ y + 2z &= -1. \end{aligned}$$

The only free variable is $z$; the corresponding parametric form is

$$\begin{aligned} x &= -5z + 1 \\ y &= -2z - 1 \\ z &= \phantom{-5}z. \end{aligned}$$

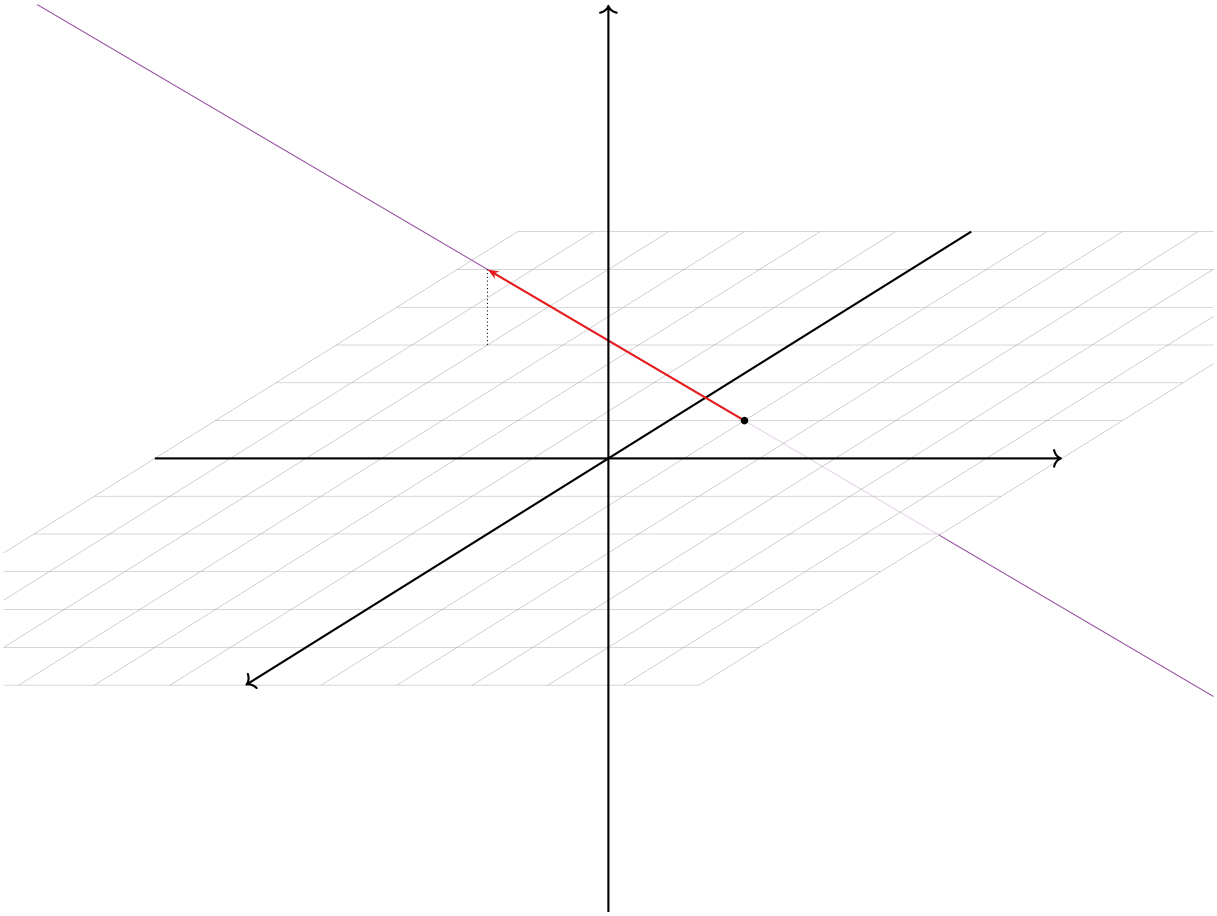
The parametric vector form is thus

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=z\begin{pmatrix}-5\\-2\\1\end{pmatrix}+\begin{pmatrix}1\\-1\\0\end{pmatrix}.$$

**e)** The solution set can be written

$$\mathrm{Span}\left\{\begin{pmatrix}-5\\-2\\1\end{pmatrix}\right\}+\begin{pmatrix}1\\-1\\0\end{pmatrix}.$$

This the line through $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ that is parallel to $\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$.



## Problem 3.

Consider the following vectors:

$$v_1 = \begin{pmatrix} 17 \\ -3 \\ 24 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 7/2 \\ 0 \\ \pi \end{pmatrix}.$$

**a)** [4 points] Describe Span$\{v_1, v_2\}$ geometrically: "it is a $\boxed{\phantom{xxxxxx}}$ in $\mathbf{R}^{\boxed{\phantom{x}}}$."

**b)** [6 points] Find a matrix $A$ with three rows, with the property that the matrix equation $Ax = b$ is consistent if and only if $b$ is in Span$\{v_1, v_2\}$.

## Solution.

**a)** Since $v_1$ and $v_2$ are noncollinear vectors in $\mathbf{R}^3$, they span a plane in $\mathbf{R}^3$.

**b)** $A = \begin{pmatrix} 17 & 7/2 \\ -3 & 0 \\ 24 & \pi \end{pmatrix}$

## Problem 4.

[5 points each]

**a)** Let $v_1 = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 7 \\ 0 \\ 1 \end{pmatrix}$, and $v_3 = \begin{pmatrix} -1 \\ -5 \\ 12 \\ 4 \end{pmatrix}$. Is $\{v_1, v_2, v_3\}$ linearly independent? If your answer is yes, justify why. If your answer is no, give a linear dependence relation for $v_1$, $v_2$, and $v_3$.

**b)** Find a vector $w$ in $\mathbf{R}^3$ that is not in $\mathrm{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$. Is the set $\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, w\right\}$ linearly independent? Justify your answer.

## Solution.

**a)** We form an augmented matrix and row reduce:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 3 & 7 & -5 & 0 \\ 4 & 0 & 12 & 0 \\ 2 & 1 & 4 & 0 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This matrix only has two pivots to the left of the augment, so $\{v_1, v_2, v_3\}$ is not linearly independent. From the augmented matrix's RREF, we see that the equation

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$$

has $x_3$ as a free variable and its solution set is

$$x_1 = -3x_3 \qquad x_2 = 2x_3 \qquad x_3 = x_3 \quad (x_3 \text{ real}).$$

For $x_3 = 1$ we get

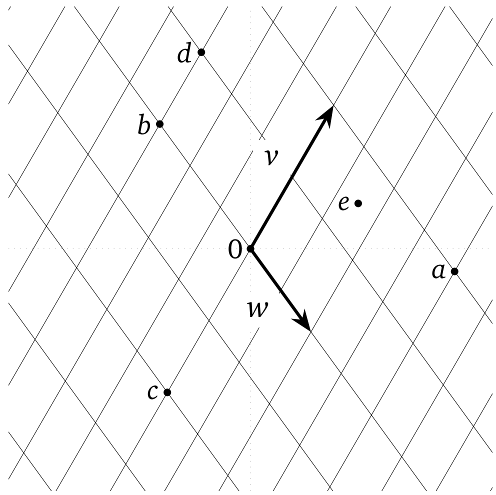
$$-3v_1 + 2v_2 + v_3 = 0.$$

**b)** The vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ span the $xz$-plane, so $w = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ works, for example. Since the first two vectors span a plane and $w$ is not in that plane, the set $\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, w\right\}$ is linearly independent by the increasing span criterion.

# Problem 5.

[5 points each]

Consider the following picture of two vectors $v, w$:




**a)** For each of the labeled points, estimate the coefficients $x, y$ such that the linear combination $xv + yw$ is the vector ending at that point.

$$\underline{\qquad}\, v + \underline{\qquad}\, w = a$$

$$\underline{\qquad}\, v + \underline{\qquad}\, w = b$$

$$\underline{\qquad}\, v + \underline{\qquad}\, w = c$$

$$\underline{\qquad}\, v + \underline{\qquad}\, w = d$$

$$\underline{\qquad}\, v + \underline{\qquad}\, w = e$$

**b)** Find two vectors $p, q$ in $\mathbf{R}^2$ such that *none* of the points $a, b, c, d, e$ is in $\mathrm{Span}\{p, q\}$.

You needn't show your work in this problem.

## Solution.

**a)** As you can tell from the grid, you reach $a$ by following $v$ once then $w$ twice. Hence $a = v + 2w$. Similarly, $b = 0v - \frac{3}{2}w$, $c = -v + 0w$, $d = \frac{1}{2}v - \frac{3}{2}w$, and $e = \frac{3}{4}v + \frac{3}{4}w$.

**b)** None of the vectors $a, b, c, d, e$ is contained in the $x$-axis. Therefore they are not contained in

$$\mathrm{Span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}\right\}.$$

[Scratch work]