Supplemental problems: Chapter 4, Determinants

1. If A is an $n \times n$ matrix, is it necessarily true that $\det(-A) = -\det(A)$? Justify your answer.

Solution.

No. Since $\det(cA) = c^n \det(A)$, we see $\det(-A) = (-1)^n \det(A)$, so $\det(-A) = \det(A)$ if n is even, and $\det(-A) = -\det(A)$ if n is odd.

2. Let A be an $n \times n$ matrix.

a) Using cofactor expansion, explain why $\det(A) = 0$ if A has a row or a column of zeros.

b) Using cofactor expansion, explain why $\det(A) = 0$ if A has adjacent identical columns.

Solution.

a) If A has zeros for all entries in row i (so $a_{i1} = a_{i2} = \cdots = a_{in} = 0$), then the cofactor expansion along row i is

$$\det(A) \;=\; a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \;=\; 0 \cdot C_{i1} + 0 \cdot C_{i2} + \cdots + 0 \cdot C_{in} = 0.$$

Similarly, if A has zeros for all entries in column j, then the cofactor expansion along column j is the sum of a bunch of zeros and is thus 0.

b) If A has identical adjacent columns, then the cofactor expansions will be identical except that one expansion's terms for $\det(A)$ will have plus signs where the other expansion's terms for $\det(A)$ have minus signs (due to the $(-1)^{\text{power}}$ factors) and vice versa.

Therefore, $\det(A) = -\det(A)$, so $\det A = 0$.

3. Find the volume of the parallelepiped in $\mathbf{R}^4$ naturally determined by the vectors

$$\begin{pmatrix} 4 \\ 1 \\ 3 \\ 8 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 7 \\ 0 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 5 \\ -5 \\ 0 \\ 7 \end{pmatrix}.$$

Solution.

We put the vectors as columns of a matrix A and find $|\det(A)|$. For this, we expand $\det(A)$ along the third row because it only has one nonzero entry.

$$\det(A) = 3(-1)^{3+1} \cdot \det\begin{pmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 7 \end{pmatrix} = 3 \cdot 5(-1)^{1+3} \det\begin{pmatrix} 7 & 2 \\ 3 & 1 \end{pmatrix} = 3(5)(1)(7-6) = 15.$$

(In the second step, we used the cofactor expansion along the first row since it had only one nonzero entry.)

The volume is $|\det(A)| = |15| = 15$.

4. Let $A = \begin{pmatrix} -1 & 1 \\ 1 & 7 \end{pmatrix}$, and define a transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = Ax$. Find the area of $T(S)$, if S is a triangle in $\mathbf{R}^2$ with area 2.

Solution.

$|\det(A)|\mathrm{Vol}(S) = |-7-1| \cdot 2 = 16.$

5. Let

$$A = \begin{pmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 5 & 4 \\ 1 & -1 & -3 & 0 \\ -1 & 0 & 5 & 4 \\ 3 & -3 & -2 & 5 \end{pmatrix}$$

a) Compute $\det(A)$.

b) Compute $\det(B)$.

c) Compute $\det(AB)$.

d) Compute $\det(A^2B^{-1}AB^2)$.

Solution.

a) Cofactor expansion would take some time, since the matrix has almost no zero entries. We use row reduction below, where r counts the row swaps and s measures the scaling factors.

$$\begin{pmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{pmatrix} \xrightarrow{R_1 = \frac{R_1}{2}} \begin{pmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{pmatrix} \; (r = 0,\ s = \frac{1}{2})$$

$$\xrightarrow[R_3 = R_3 + 3R_1,\ R_4 = R_4 - R_1]{R_2 = R_2 - 3R_1} \begin{pmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{pmatrix} \; (r = 0,\ s = \frac{1}{2})$$

$$\xrightarrow{R_3 = R_3 + 4R_2} \begin{pmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{pmatrix} \; (r = 0,\ s = \frac{1}{2})$$

$$\xrightarrow{R_4 = R_4 - \frac{R_3}{2}} \begin{pmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \; (r = 0,\ s = \frac{1}{2})$$

$$\det(A) = (-1)^0 \frac{1 \cdot 3 \cdot (-6) \cdot 1}{1/2} = -36.$$

b) This is a complicated matrix without a lot of zeros, so we compute the determinant by row reduction. After one row swap and several row replacements, we reduce to the matrix $\begin{pmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & -3 \end{pmatrix}$. The determinant of this matrix is -21, so the determinant of the original matrix is 21.

c) $\det(AB) = \det(A)\det(B) = (-36)(21) = -756$.

d) $\det(A^2B^{-1}AB^2) = \det(A)^2\det(B)^{-1}\det(A)\det(B)^2 = \det(A)^3\det(B) = (-36)^3(21) = -979{,}776$.

6. If A is a 3×3 matrix and $\det(A) = 1$, what is $\det(-2A)$?

Solution.

By determinant properties, scaling one row by c multiplies the determinant by c. When we take cA for an $n \times n$ matrix A, we are multiplying *each* row by c. This multiplies the determinant by c a total of n times.

Thus, if A is $n \times n$, then $\det(cA) = c^n \det(A)$. Here $n = 3$, so

$$\det(-2A) = (-2)^3\det(A) = -8\det(A) = -8.$$

7. **a)** Is there a real 2×2 matrix A that satisfies $A^4 = -I_2$? Either write such an A, or show that no such A exists.
(hint: think geometrically! The matrix $-I_2$ represents rotation by π radians).

b) Is there a real 3×3 matrix A that satisfies $A^4 = -I_3$? Either write such an A, or show that no such A exists.

Solution.

a) Yes. Just take A to be the matrix of counterclockwise rotation by $\frac{\pi}{4}$ radians:

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then A^2 gives rotation c.c. by $\frac{\pi}{2}$ radians, A^3 gives rotation c.c. by $\frac{3\pi}{4}$ radians, and A^4 gives rotation c.c. by π radians, which has matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_2$.

b) No. If $A^4 = -I$ then

$$[\det(A)]^4 = \det(A^4) = \det(-I) = (-1)^3 = -1.$$

In other words, if $A^4 = -I$ then $[\det(A)]^4 = -1$, which is impossible since $\det(A)$ is a real number.

Similarly, $A^4 = -I$ is impossible if A is 5×5, 7×7, etc.

Supplemental problems: §5.1

1. True or false. Answer true if the statement is always true. Otherwise, answer false.

a) If A and B are $n \times n$ matrices and A is row equivalent to B, then A and B have the same eigenvalues.

b) If A is an $n \times n$ matrix and its eigenvectors form a basis for $\mathbf{R}^n$, then A is invertible.

c) If 0 is an eigenvalue of the $n \times n$ matrix A, then $\operatorname{rank}(A) < n$.

d) The diagonal entries of an $n \times n$ matrix A are its eigenvalues.

e) If A is invertible and 2 is an eigenvalue of A, then $\frac{1}{2}$ is an eigenvalue of A^{-1}.

f) If $\det(A) = 0$, then 0 is an eigenvalue of A.

Solution.

a) False. For instance, the matrices $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are row equivalent, but have different eigenvalues.

b) False. For example, the zero matrix is not invertible but its eigenvectors form a basis for $\mathbf{R}^n$.

c) True. If $\lambda = 0$ is an eigenvalue of A then A is not invertible so its associated transformation $T(x) = Ax$ is not onto, hence $\operatorname{rank}(A) < n$.

d) False. This is true if A is triangular, but not in general.
For example, if $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ then the diagonal entries are 2 and 0 but the only eigenvalue is $\lambda = 1$, since solving the characteristic equation gives us

$$(2-\lambda)(-\lambda)-(1)(-1)=0 \qquad \lambda^2-2\lambda+1=0 \qquad (\lambda-1)^2=0 \qquad \lambda=1.$$

e) True. Let v be an eigenvector corresponding to the eigenvalue 2.

$$Av = 2v \implies A^{-1}Av = A^{-1}(2v) \implies v = 2A^{-1}v \implies \frac{1}{2}v = A^{-1}v.$$

Therefore, v is an eigenvector of A^{-1} corresponding to the eigenvalue $\frac{1}{2}$.

f) True. If $\det(A) = 0$ then A is not invertible, so $Av = 0v$ has a nontrivial solution.

2. In this problem, you need not explain your answers; just circle the correct one(s).

Let A be an $n \times n$ matrix.

a) Which **one** of the following statements is correct?

1. An eigenvector of A is a vector v such that $Av = \lambda v$ for a nonzero scalar λ.

2. An eigenvector of A is a nonzero vector v such that $Av = \lambda v$ for a scalar λ.

3. An eigenvector of A is a nonzero scalar λ such that $Av = \lambda v$ for some vector v.

4. An eigenvector of A is a nonzero vector v such that $Av = \lambda v$ for a nonzero scalar λ.

b) Which **one** of the following statements is **not** correct?

1. An eigenvalue of A is a scalar λ such that $A - \lambda I$ is not invertible.

2. An eigenvalue of A is a scalar λ such that $(A - \lambda I)v = 0$ has a solution.

3. An eigenvalue of A is a scalar λ such that $Av = \lambda v$ for a nonzero vector v.

4. An eigenvalue of A is a scalar λ such that $\det(A - \lambda I) = 0$.

Solution.

a) Statement 2 is correct: an eigenvector must be nonzero, but its eigenvalue may be zero.

b) Statement 2 is incorrect: the solution v must be nontrivial.

3. Find a basis $\mathcal{B}$ for the (-1)-eigenspace of $Z = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix}$

Solution.

For $\lambda = -1$, we find $\mathrm{Nul}(Z - \lambda I)$.

$$\left(Z - \lambda I \mid 0\right) = \left(Z + I \mid 0\right) = \left(\begin{array}{ccc|c} 3 & 3 & 1 & 0 \\ 3 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Therefore, $x = -y$, $y = y$, and $z = 0$, so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

A basis is $\mathcal{B} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$. We can check to ensure $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector with corresponding eigenvalue -1:

$$Z \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2+3 \\ -3+2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

4. Suppose A is an $n \times n$ matrix satisfying $A^2 = 0$. Find all eigenvalues of A. Justify your answer.

Solution.

If λ is an eigenvalue of A and $v \neq 0$ is a corresponding eigenvector, then

$$Av = \lambda v \implies A(Av) = A\lambda v \implies A^2 v = \lambda(Av) \implies 0 = \lambda(\lambda v) \implies 0 = \lambda^2 v.$$

Since $v \neq 0$ this means $\lambda^2 = 0$, so $\lambda = 0$. This shows that 0 is the only possible eigenvalue of A.

On the other hand, $\det(A) = 0$ since $(\det(A))^2 = \det(A^2) = \det(0) = 0$, so 0 must be an eigenvalue of A. Therefore, the only eigenvalue of A is 0.

5. Match the statements (i)-(v) with the corresponding statements (a)-(e). All matrices are 3×3. There is a unique correspondence. Justify the correspondences in words.

(i) $Ax = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ has a unique solution.
(ii) The transformation $T(v) = Av$ fixes a nonzero vector.
(iii) A is obtained from B by subtracting the third row of B from the first row of B.
(iv) The columns of A and B are the same; except that the first, second and third columns of A are respectively the first, third, and second columns of B.
(v) The columns of A, when added, give the zero vector.

(a) 0 is an eigenvalue of A.
(b) A is invertible.
(c) $\det(A) = \det(B)$
(d) $\det(A) = -\det(B)$
(e) 1 is an eigenvalue of A.

Solution.

(i) matches with (b).
(ii) matches with (e).
(iii) matches with (c).
(iv) matches with (d).
(v) matches with (a).