Supplemental problems: §2.3, §2.4

Problems 1 and 2 use the same widgets and gizmos class from a worksheet. The professor in your widgets and gizmos class is trying to decide between three different grading schemes for computing your final course grade. The schemes are based on homework (HW), quiz grades (Q), midterms (M), and a final exam (F). The three schemes can be described by the following matrix A:

$$\begin{array}{c} \\ \text{Scheme 1} \\ \text{Scheme 2} \\ \text{Scheme 3} \end{array} \begin{array}{c} \begin{array}{cccc} \text{HW} & \text{Q} & \text{M} & \text{F} \end{array} \\ \begin{pmatrix} 0.1 & 0.1 & 0.5 & 0.3 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.1 & 0.1 & 0.6 & 0.2 \end{pmatrix} \end{array}$$

1. Suppose that you end up with averages of 90% on the homework, 90% on quizzes, 85% on midterms, and a 95% score on the final exam. Determine which grading scheme leaves you with the highest overall course grade.

2. Suppose that you have a score of x_1 on homework, x_2 on quizzes, x_3 on midterms, and x_4 on the final, with potential final course grades of b_1, b_2, b_3.

a) In a worksheet, you wrote the matrix equation $Ax = b$ to relate your final grades to your scores. Keeping $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ as a general vector, write the augmented matrix $(A \mid b)$.

b) Row reduce this matrix until you reach reduced row echelon form.

c) Looking at the final matrix in (b), what equation in terms of b_1, b_2, b_3 must be satisfied in order for $Ax = b$ to have a solution?

d) The answer to (c) also defines the span of the columns of A. Describe the span geometrically.

e) Solve the equation in (c) for b_1. Looking at this equation, is it possible for b_1 to be the largest of b_1, b_2, b_3? In other words, is it ever possible for the grade under Scheme 1 to be the highest of the three final course grades? Why or why not? Which scheme would you argue for?

3. True or false. If the statement is *ever* false, answer false. Justify your answer.

a) A matrix equation $Ax = b$ is consistent if A has a pivot in every column.

b) If an $m \times n$ matrix A has fewer than n pivots and b is in $\mathbf{R}^m$, then $Ax = b$ has infinitely many solutions.

c) Suppose A is a 3×3 matrix and there is a vector y in $\mathbf{R}^3$ so that $Ax = y$ does not have a solution. Is it possible that there is a z in $\mathbf{R}^3$ so that the equation $Ax = z$ has a *unique* solution? Justify your answer.

d) There is a matrix A and a nonzero vector b so that the solution set of $Ax = b$ is a plane through the origin.

e) Suppose A is an $m \times n$ matrix and b is in $\mathbf{R}^m$. If the columns of A span $\mathbf{R}^m$, then $Ax = b$ must be consistent.

f) If $Ax = b$ is consistent, then the solution set is a span.

4. For each of the following, give an example if it is possible. If it is not possible, justify why there is no such example.

a) A 3×4 matrix A in RREF with 2 pivot columns, so that for some vector b, the system $Ax = b$ has exactly three free variables.

b) A homogeneous linear system with no solution.

c) A 5×3 matrix in RREF such that $Ax = 0$ has a non-trivial solution.

5. Suppose the solution set of a certain system of linear equations is given by

$$x_1 = 9 + 8x_4, \qquad x_2 = -9 - 14x_4, \qquad x_3 = 1 + 2x_4, \qquad x_4 = x_4 \ \ (x_4 \text{ free}).$$

Write the solution set in parametric vector form. Describe the set geometrically.

6. a) What best describes $\text{Span}\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}2\\3\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$? Justify your answer.

(I) It is a plane through the origin.

(II) It is three lines through the origin.

(III) It is all of $\mathbf{R}^3$.

(IV) It is a plane, plus the line through the origin and the vector $\begin{pmatrix}0\\1\\0\end{pmatrix}$.

b) Does $\text{Span}\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\2\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\3\end{pmatrix}\right\} = \mathbf{R}^3$? If yes, justify your answer. If not, write a vector in $\mathbf{R}^3$ which is not in the span of those three vectors.

7. Let $A = \begin{pmatrix}5 & -5 & 10\\3 & -3 & 6\end{pmatrix}$. Draw the column span of A.

8. Consider the following consistent system of linear equations.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= -2 \\ 3x_1 + 4x_2 + 5x_3 + 6x_4 &= -2 \\ 5x_1 + 6x_2 + 7x_3 + 8x_4 &= -2 \end{aligned}$$

a) Find the parametric vector form for the general solution.

b) Find the parametric vector form of the corresponding *homogeneous* equations.

Supplemental problems: §2.5

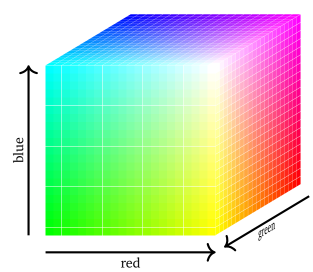
1. Justify why each of the following true statements can be checked without row reduction.

a) $\left\{\begin{pmatrix}3\\3\\4\end{pmatrix}, \begin{pmatrix}0\\0\\\pi\end{pmatrix}, \begin{pmatrix}0\\\sqrt{2}\\0\end{pmatrix}\right\}$ is linearly independent.

b) $\left\{\begin{pmatrix}3\\3\\4\end{pmatrix}, \begin{pmatrix}0\\10\\20\end{pmatrix}, \begin{pmatrix}0\\5\\7\end{pmatrix}\right\}$ is linearly independent.

c) $\left\{\begin{pmatrix}3\\3\\4\end{pmatrix}, \begin{pmatrix}0\\10\\20\end{pmatrix}, \begin{pmatrix}0\\5\\7\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ is linearly dependent.

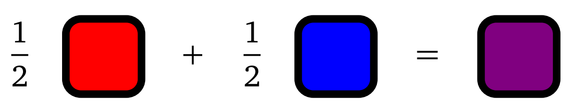
2. Every color on my computer monitor is a vector in $\mathbf{R}^3$ with coordinates between 0 and 255, inclusive. The coordinates correspond to the amount of red, green, and blue in the color.



Given colors $v_1, v_2, \ldots, v_p$, we can form a "weighted average" of these colors by making a linear combination

$$v = c_1v_1 + c_2v_2 + \cdots + c_pv_p$$

with $c_1 + c_2 + \cdots + c_p = 1$. Example:

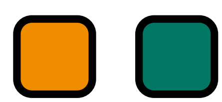


Consider the colors on the right. For which h is

$$\left\{\begin{pmatrix}180\\50\\200\end{pmatrix}, \begin{pmatrix}100\\150\\100\end{pmatrix}, \begin{pmatrix}116\\130\\h\end{pmatrix}\right\}$$

linearly dependent? What does that say about the corresponding color?

$\begin{pmatrix}180\\50\\200\end{pmatrix}$ $\begin{pmatrix}100\\150\\100\end{pmatrix}$



$h =$