Recall: an $n \times n$ matrix A is **diagonalizable** if it is similar to a diagonal matrix:

$$A = CDC^{-1} \quad \text{for} \quad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

It is easy to take powers of diagonalizable matrices:

$$A^i = CD^iC^{-1} = C \begin{pmatrix} \lambda_1^i & 0 & \cdots & 0 \\ 0 & \lambda_2^i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^i \end{pmatrix} C^{-1}.$$

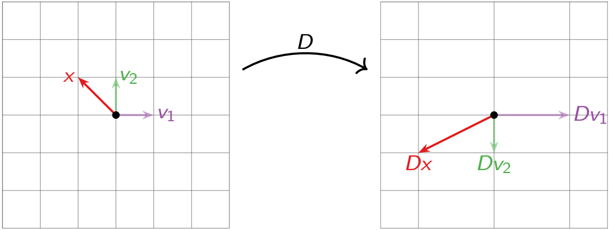
We begin today by discussing the *geometry* of diagonalizable matrices.

Geometry of Diagonal Matrices

A diagonal matrix $D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ just scales the coordinate axes:

$$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This is easy to visualize:



$$x = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \Longrightarrow \quad Dx = \begin{pmatrix} -2 \\ -1 \end{pmatrix}.$$

Geometry of Diagonal*izable* Matrices

We had this example last time: $A = CDC^{-1}$ for

$$A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

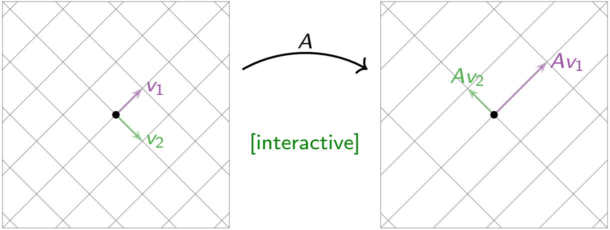
The eigenvectors of A are $v_1 = \binom{1}{1}$ and $v_2 = \binom{1}{-1}$ with eigenvalues 2 and -1.

The eigenvectors form a *basis* for $\mathbf{R}^2$ because they're linearly independent.

Any vector can be written as a linear combination of basis vectors:

$$x = a_1 v_1 + a_2 v_2 \implies Ax = A(a_1 v_1 + a_2 v_2) = a_1 Av_1 + a_2 Av_2 = 2a_1 v_1 - a_2 v_2.$$

Conclusion: A scales the "v_1-direction" by 2 and the "v_2-direction" by -1.

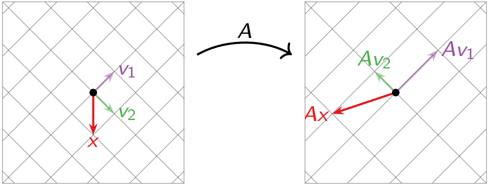


Geometry of Diagonal*izable* Matrices

Continued

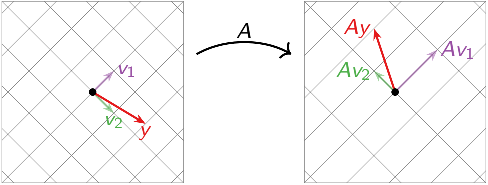
Example: $x = \binom{0}{-2} = -1v_1 + 1v_2$

$$Ax = -1Av_1 + 1Av_2 = -2v_1 + -1v_2 = -2\begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}-3\\-1\end{pmatrix}.$$



Example: $y = \frac{1}{2}\binom{5}{-3} = \frac{1}{2}v_1 + 2v_2$

$$Ay = \frac{1}{2}Av_1 + 2Av_2 = 1v_1 + -2v_2 = \begin{pmatrix}1\\1\end{pmatrix} - 2\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}-1\\3\end{pmatrix}.$$



Dynamics of Diagonalizable Matrices

We motivated diagonalization by taking powers:

$$A^i = CD^iC^{-1} = C \begin{pmatrix} \lambda_1^i & 0 & \cdots & 0 \\ 0 & \lambda_2^i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^i \end{pmatrix} C^{-1}.$$

This lets us compute powers of matrices easily. How to visualize this?

$$A^n v = A(A(A\cdots(Av))\cdots)$$

Multiplying a vector v by A^n means repeatedly multiplying by A.

Dynamics of Diagonalizable Matrices

Example

$$A = \frac{1}{10}\begin{pmatrix} 11 & 6 \\ 9 & 14 \end{pmatrix} = CDC^{-1} \quad \text{for} \quad C = \begin{pmatrix} 2/3 & 1 \\ -1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Eigenvectors of A are $v_1 = \binom{2/3}{-1}$ and $v_2 = \binom{1}{1}$ with eigenvalues 2 and 1/2.

$$\begin{aligned} A(a_1v_1 + a_2v_2) &= 2a_1v_1 + \frac{1}{2}a_2v_2 \\ A^2(a_1v_1 + a_2v_2) &= 4a_1v_1 + \frac{1}{4}a_2v_2 \\ A^3(a_1v_1 + a_2v_2) &= 8a_1v_1 + \frac{1}{8}a_2v_2 \\ &\vdots \\ A^n(a_1v_1 + a_2v_2) &= 2^n a_1v_1 + \frac{1}{2^n}a_2v_2 \end{aligned}$$

What does repeated application of A do geometrically?

It makes the "v_1-coordinate" very big, and the "v_2-coordinate" very small.

[interactive]

$$A = \frac{1}{6}\begin{pmatrix} 5 & -1 \\ -2 & 4 \end{pmatrix} = CDC^{-1} \quad \text{for} \quad C = \begin{pmatrix} -1 & 1/2 \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Eigenvectors of A are $v_1 = \binom{-1}{1}$ and $v_2 = \binom{1/2}{1}$ with eigenvalues 1 and 1/2.

$$\begin{aligned} A(a_1 v_1 + a_2 v_2) &= a_1 v_1 + \frac{1}{2} a_2 v_2 \\ A^2(a_1 v_1 + a_2 v_2) &= a_1 v_1 + \frac{1}{4} a_2 v_2 \\ A^3(a_1 v_1 + a_2 v_2) &= a_1 v_1 + \frac{1}{8} a_2 v_2 \\ &\vdots \\ A^n(a_1 v_1 + a_2 v_2) &= a_1 v_1 + \frac{1}{2^n} a_2 v_2 \end{aligned}$$

What does repeated application of A do geometrically?

It "sucks everything into the 1-eigenspace."

[interactive]

$$A = \frac{1}{30}\begin{pmatrix} 12 & 2 \\ 3 & 13 \end{pmatrix} = CDC^{-1} \quad \text{for} \quad C = \begin{pmatrix} 2/3 & -1 \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix}.$$

Poll

What does repeated application of A do geometrically?

A. Sucks all vectors into a line.

B. Sucks all vectors into the origin.

C. Shoots all vectors away from a line.

D. Shoots all vectors away from the origin.

B. Since both eigenvalues are less than 1, the matrix A scales *both* directions towards the origin.

[interactive]

Section 5.5

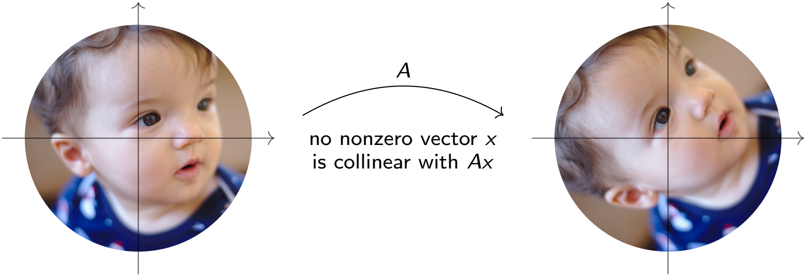
Complex Eigenvalues

A Matrix with No Eigenvectors

Consider the matrix for the linear transformation for rotation by $\pi/4$ in the plane. The matrix is:

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

This matrix has no eigenvectors, as you can see geometrically: [interactive]



or algebraically:

$$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\,\lambda + \det(A) = \lambda^2 - \sqrt{2}\,\lambda + 1 \implies \lambda = \frac{\sqrt{2} \pm \sqrt{-2}}{2}.$$

Complex Numbers

It makes us sad that -1 has no square root. If it did, then $\sqrt{-2} = \sqrt{2} \cdot \sqrt{-1}$.

Mathematician's solution: we're just not using enough numbers! We're going to declare by *fiat* that there exists a square root of -1.

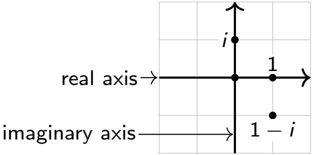
Definition
The number i is defined such that $i^2 = -1$.

Once we have i, we have to allow numbers like $a + bi$ for real numbers a, b.

Definition
A *complex number* is a number of the form $a + bi$ for a, b in $\mathbf{R}$. The set of all complex numbers is denoted $\mathbf{C}$.

Note $\mathbf{R}$ is contained in $\mathbf{C}$: they're the numbers $a + 0i$.

We can identify $\mathbf{C}$ with $\mathbf{R}^2$ by $a + bi \longleftrightarrow \binom{a}{b}$. So when we draw a picture of $\mathbf{C}$, we draw the plane:



Operations on Complex Numbers

Addition: $(2-3i)+(-1+i)=1-2i$.

Multiplication: $(2-3i)(-1+i)=2(-1)+2i+3i-3i^2=-2+5i+3=1+5i$.

Complex conjugation: $\overline{a+bi}=a-bi$ is the **complex conjugate** of $a+bi$.
Check: $\overline{z+w}=\overline{z}+\overline{w}$ and $\overline{zw}=\overline{z}\cdot\overline{w}$.

Absolute value: $|a+bi|=\sqrt{a^2+b^2}$. This is a *real* number.
Note: $(a+bi)(\overline{a+bi})=(a+bi)(a-bi)=a^2-(bi)^2=a^2+b^2$. So $|z|=\sqrt{z\overline{z}}$.
Check: $|zw|=|z|\cdot|w|$.

Division by a nonzero real number: $\dfrac{a+bi}{c}=\dfrac{a}{c}+\dfrac{b}{c}i$.

Division by a nonzero complex number: $\dfrac{z}{w}=\dfrac{z\overline{w}}{w\overline{w}}=\dfrac{z\overline{w}}{|w|^2}$.

Example:

$$\frac{1+i}{1-i}=\frac{(1+i)^2}{1^2+(-1)^2}=\frac{1+2i+i^2}{2}=i.$$

Real and imaginary part: $\mathrm{Re}(a+bi)=a \qquad \mathrm{Im}(a+bi)=b$.

The Fundamental Theorem of Algebra

The whole point of using complex numbers is to solve polynomial equations. It turns out that they are enough to find all solutions of all polynomial equations:

Fundamental Theorem of Algebra

Every polynomial of degree n has exactly n complex roots, counted with multiplicity.

Equivalently, if $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree n, then

$$f(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)$$

for (not necessarily distinct) complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Important

If f is a polynomial with *real* coefficients, and if λ is a complex root of f, then so is $\overline{\lambda}$:

$$\begin{aligned} 0 = \overline{f(\lambda)} &= \overline{\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0} \\ &= \overline{\lambda}^n + a_{n-1}\overline{\lambda}^{n-1} + \cdots + a_1\overline{\lambda} + a_0 = f\big(\overline{\lambda}\big). \end{aligned}$$

Therefore complex roots of real polynomials come in *conjugate pairs*.

Degree 2: The quadratic formula gives you the (real or complex) roots of any degree-2 polynomial:

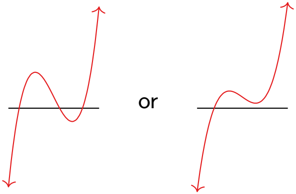
$$f(x) = x^2 + bx + c \implies x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

For instance, if $f(\lambda) = \lambda^2 - \sqrt{2}\lambda + 1$ then

$$\lambda = \frac{\sqrt{2} \pm \sqrt{-2}}{2} = \frac{\sqrt{2}}{2}(1 \pm i) = \frac{1 \pm i}{\sqrt{2}}.$$

Note the roots are complex conjugates if b, c are real.

Degree 3: A real cubic polynomial has either three real roots, or one real root and a conjugate pair of complex roots. The graph looks like:



respectively.

A Matrix *with* an Eigenvector

Every matrix is guaranteed to have *complex* eigenvalues and eigenvectors. Using rotation by $\pi/4$ from before:

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{has eigenvalues} \quad \lambda = \frac{1 \pm i}{\sqrt{2}}.$$

Let's compute an eigenvector for $\lambda = (1+i)/\sqrt{2}$:

$$A - \lambda I = \frac{1}{\sqrt{2}}\begin{pmatrix} 1-(1+i) & -1 \\ 1 & 1-(1+i) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}.$$

The second row is i times the first, so we row reduce:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \rightsquigarrow \frac{1}{\sqrt{2}}\begin{pmatrix} -i & -1 \\ 0 & 0 \end{pmatrix} \xrightarrow{\text{divide by } -i/\sqrt{2}} \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}.$$

The parametric form is $x = iy$, so an eigenvector is $\begin{pmatrix} i \\ 1 \end{pmatrix}$. So is *any* nonzero complex scalar multiple of $\begin{pmatrix} i \\ 1 \end{pmatrix}$, for example $(-i/\sqrt{2})\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$.

A similar computation shows that an eigenvector for $\lambda = (1-i)/\sqrt{2}$ is $\begin{pmatrix} -i \\ 1 \end{pmatrix}$.

So is $i\begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ (you can scale by *complex* numbers).

Conjugate Eigenvectors

For $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$,

$$\text{the eigenvalue } \frac{1+i}{\sqrt{2}} \text{ has eigenvector } \begin{pmatrix} i \\ 1 \end{pmatrix}.$$
$$\text{the eigenvalue } \frac{1-i}{\sqrt{2}} \text{ has eigenvector } \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Do you notice a pattern?

Fact

Let A be a real square matrix. If λ is a complex eigenvalue with eigenvector v, then $\overline{\lambda}$ is an eigenvalue with eigenvector $\overline{v}$.

Why?

$$Av = \lambda \implies A\overline{v} = \overline{Av} = \overline{\lambda v} = \overline{\lambda}\,\overline{v}.$$

> Both eigenvalues and eigenvectors of real square matrices occur in conjugate pairs.

2×2 eigenvector trick

Suppose A is a 2×2 matrix and λ is any eigenvalue of A. Then

$$A - \lambda I_2 = \begin{pmatrix} z & w \\ (*) & (*) \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} -w \\ z \end{pmatrix}$$

is an eigenvector of A corresponding to the eigenvalue λ.

In the previous example, $\lambda = \dfrac{1+i}{\sqrt{2}}$ was an eigenvalue of $A = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and

$$A - \lambda I = \begin{pmatrix} -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} z & w \\ (*) & (*) \end{pmatrix}.$$

So an eigenvector of A corresponding to λ is

$$v = \begin{pmatrix} -w \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \\ -\frac{i}{\sqrt{2}} \end{pmatrix}.$$

This was much faster than doing the full 2×2 row reduction in the previous example, and it agrees with our answer.

A 3×3 Example

Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The characteristic polynomial is

$$f(\lambda) = \det \begin{pmatrix} \frac{4}{5} - \lambda & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} - \lambda & 0 \\ 0 & 0 & 2 - \lambda \end{pmatrix} = \boxed{(2 - \lambda)} \left(\lambda^2 - \frac{8}{5}\lambda + 1 \right).$$

This factors out automatically if you expand cofactors along the third row or column

We computed the roots of this polynomial (times 5) before:

$$\lambda = \quad 2, \quad \frac{4+3i}{5}, \quad \frac{4-3i}{5}.$$

We eyeball an eigenvector with eigenvalue 2 as $(0, 0, 1)$.

A 3×3 Example

Continued

$$A = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

To find the other eigenvectors, we row reduce:

$$A - \frac{4+3i}{5} I = \begin{pmatrix} -\frac{3}{5}i & -\frac{3}{5} & 0 \\ \frac{3}{5} & -\frac{3}{5}i & 0 \\ 0 & 0 & 2 - \frac{4+3i}{5} \end{pmatrix} \xrightarrow{\text{scale rows}} \begin{pmatrix} -i & -1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The second row is i times the first:

$$\xrightarrow{\text{row replacement}} \begin{pmatrix} -i & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{divide by } -i\text{,swap}} \begin{pmatrix} 1 & -i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The parametric form is $x = iy$, $z = 0$, so an eigenvector is $\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$. Therefore, an eigenvector with conjugate eigenvalue $\frac{4-3i}{5}$ is $\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}$.

Summary

- Diagonal matrices are easy to understand geometrically.
- Diagonalizable matrices behave like diagonal matrices, except with respect to a basis of eigenvectors.
- Repeatedly multiplying by a matrix leads to fun pictures.
- One can do arithmetic with complex numbers just like real numbers: add, subtract, multiply, divide.
- An $n \times n$ matrix always exactly has complex n eigenvalues, counted with (algebraic) multiplicity.
- The complex eigenvalues and eigenvectors of a *real* matrix come in complex conjugate pairs:

$$Av = \lambda v \quad \implies \quad A\overline{v} = \overline{\lambda}\overline{v}.$$