Chapter 4

Determinants

Section 4.1

Determinants: Definition

Orientation

Recall: This course is about learning to:

- Solve the matrix equation $Ax = b$
  We've said most of what we'll say about this topic now.
- Solve the matrix equation $Ax = \lambda x$ (eigenvalue problem)
  We are now aiming at this.
- Almost solve the equation $Ax = b$
  This will happen later.

The next topic is *determinants*.

This is a completely magical function that takes a square matrix and gives you a number.

It is a very complicated function—the formula for the determinant of a 10×10 matrix has $3,628,800$ summands—so instead of writing down the formula, we'll give other ways to compute it.

Today is mostly about the *theory* of the determinant; in the next lecture we will focus on *computation*.

A Definition of Determinant

Definition

The **determinant** is a function

determinants are only for square matrices!

$$\det\colon \{n \times n \text{ matrices}\} \longrightarrow \mathbf{R}$$

with the following properties:

1. If you do a row replacement on a matrix, the determinant doesn't change.
2. If you scale a row by c, the determinant is multiplied by c.
3. If you swap two rows of a matrix, the determinant is multiplied by -1.
4. $\det(I_n) = 1$.

Example:

$$\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \xrightarrow{R_1 \longleftrightarrow R_2} \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix} \qquad \det = 7 \checkmark$$

$$\xrightarrow{R_2 = R_2 - 2R_1} \begin{pmatrix} 1 & 4 \\ 0 & -7 \end{pmatrix} \qquad \det = -7$$

$$\xrightarrow{R_2 = R_2 \div -7} \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \qquad \det = 1$$

$$\xrightarrow{R_1 = R_1 - 4R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \det = 1$$

A Definition of Determinant

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4. $\det(I_n) = 1$.

> This is a *definition* because it tells you how to compute the determinant: row reduce!

It's not at all obvious that you get the same determinant if you row reduce in two different ways, but this is magically true!

Special Cases

Special Case 1

If A has a zero row, then $\det(A) = 0$.

Why?

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow{R_2 = -R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix}$$

The determinant of the second matrix is negative the determinant of the first (property 3), so

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix} = -\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix}.$$

This implies the determinant is zero.

Special Case 2

If A is upper-triangular, then the determinant is the product of the diagonal entries:

$$\det\begin{pmatrix} a & \star & \star \\ 0 & b & \star \\ 0 & 0 & c \end{pmatrix} = abc.$$

Upper-triangular means the only nonzero entries are on or above the diagonal.

Why?

- If one of the diagonal entries is zero, then the matrix has fewer than n pivots, so the RREF has a row of zeros. (Row operations don't change whether the determinant is zero.)
- Otherwise,

$$\begin{pmatrix} a & \star & \star \\ 0 & b & \star \\ 0 & 0 & c \end{pmatrix} \xrightarrow[]{\text{scale by } a^{-1}, b^{-1}, c^{-1}} \begin{pmatrix} 1 & \star & \star \\ 0 & 1 & \star \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[]{\text{row replacements}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det = abc \quad \longleftarrow \quad \det = 1 \quad \longleftarrow \quad \det = 1$$

Theorem

Let A be a square matrix. Suppose you do some number of row operations on A to get a matrix B in row echelon form. Then

$$\det(A) = (-1)^r \frac{\text{(product of the diagonal entries of } B\text{)}}{\text{(product of the scaling factors)}},$$

where r is the number of row swaps.

Why? Since B is in REF, it is upper-triangular, so its determinant is the product of its diagonal entries. You changed the determinant by $(-1)^r$ and the product of the scaling factors when going from A to B.

Remark

This is generally the fastest way to compute a determinant of a large matrix, either by hand or by computer.

Row reduction is $O(n^3)$; cofactor expansion (next time) is $O(n!) \sim O(n^n\sqrt{n})$.

This is important in real life, when you're usually working with matrices with a gazillion columns.

Computing Determinants

Example

$$\begin{pmatrix} 0 & -7 & -4 \\ 2 & 4 & 6 \\ 3 & 7 & -1 \end{pmatrix} \xrightarrow{R_1 \longleftrightarrow R_2} \begin{pmatrix} 2 & 4 & 6 \\ 0 & -7 & -4 \\ 3 & 7 & -1 \end{pmatrix} \quad r = 1$$

$$\xrightarrow{R_1 = R_1 \div 2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -7 & -4 \\ 3 & 7 & -1 \end{pmatrix} \quad \begin{array}{l} r = 1 \\ \text{scaling factors} = \frac{1}{2} \end{array}$$

$$\xrightarrow{R_3 = R_3 - 3R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -7 & -4 \\ 0 & 1 & -10 \end{pmatrix} \quad \begin{array}{l} r = 1 \\ \text{scaling factors} = \frac{1}{2} \end{array}$$

$$\xrightarrow{R_2 \longleftrightarrow R_3} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -10 \\ 0 & -7 & -4 \end{pmatrix} \quad \begin{array}{l} r = 2 \\ \text{scaling factors} = \frac{1}{2} \end{array}$$

$$\xrightarrow{R_3 = R_3 + 7R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -10 \\ 0 & 0 & -74 \end{pmatrix} \quad \begin{array}{l} r = 2 \\ \text{scaling factors} = \frac{1}{2} \end{array}$$

$$\Longrightarrow \det \begin{pmatrix} 0 & -7 & -4 \\ 1 & 4 & 6 \\ 3 & 7 & -1 \end{pmatrix} = (-1)^2 \frac{1 \cdot 1 \cdot -74}{1/2} = -148.$$

Computing Determinants

2×2 Example

Let's compute the determinant of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, a general 2×2 matrix.

- If $a = 0$, then

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} = -\det \begin{pmatrix} c & d \\ 0 & b \end{pmatrix} = -bc.$$

- Otherwise,

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \cdot \det \begin{pmatrix} 1 & b/a \\ c & d \end{pmatrix} = a \cdot \det \begin{pmatrix} 1 & b/a \\ 0 & d - c \cdot b/a \end{pmatrix}$$
$$= a \cdot 1 \cdot (d - bc/a) = ad - bc.$$

In both cases, the determinant magically turns out to be

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Poll

Poll

True or false:

(a) Row operations can change the determinant of a matrix.
(b) Row operations can change whether the determinant of a matrix is equal to zero.

(a) **True:** scaling and row swaps change the determinant by a nonzero number and by -1, respectively.
(b) **False:** all row operations multiply the determinant by a *nonzero* number.

Determinants and Invertibility

Theorem

A square matrix A is invertible if and only if $\det(A)$ is nonzero.

Why?

- If A is invertible, then its reduced row echelon form is the identity matrix, which has determinant equal to 1.
- If A is not invertible, then its reduced row echelon form has a zero row, hence has zero determinant.
- Doing row operations doesn't change whether the determinant is zero.

Determinants and Products

Theorem

If A and B are two $n \times n$ matrices, then

$$\det(AB) = \det(A) \cdot \det(B).$$

Why? If B is invertible, we can define

$$f(A) = \frac{\det(AB)}{\det(B)}.$$

Note $f(I_n) = \det(I_n B)/\det(B) = 1$. Check that f satisfies the same properties as det with respect to row operations. So

$$\det(A) = f(A) = \frac{\det(AB)}{\det(B)} \implies \det(AB) = \det(A)\det(B).$$

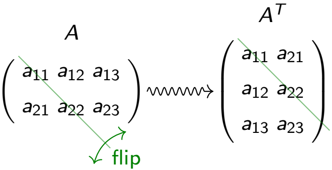
What about if B is not invertible?

Theorem

If A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Why? $I_n = AB \implies 1 = \det(I_n) = \det(AB) = \det(A)\det(B)$.

Recall: The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose rows are the columns of A. In other words, the ij entry of A^T is a_{ji}.



Determinants and Transposes

Theorem

If A is a square matrix, then

$$\det(A) = \det(A^T),$$

where A^T is the transpose of A.

Example: $\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \det\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

As a consequence, det behaves the same way with respect to *column* operations as row operations.

Corollary

an immediate consequence of a theorem

If A has a zero column, then $\det(A) = 0$.

Corollary

The determinant of a *lower*-triangular matrix is the product of the diagonal entries.

(The transpose of a lower-triangular matrix is upper-triangular.)

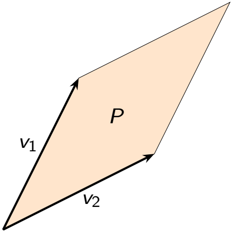
Section 4.3

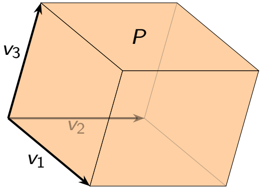
Determinants and Volumes

Now we discuss a completely different description of (the absolute value of) the determinant, in terms of volumes.

This is a crucial component of the change-of-variables formula in multivariable calculus.

The columns $v_1, v_2, \ldots, v_n$ of an $n \times n$ matrix A give you n vectors in $\mathbf{R}^n$. These determine a **parallelepiped** P.





Theorem

Let A be an $n \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, and let P be the parallelepiped determined by A. Then

$$(\text{volume of } P) = |\det(A)|.$$

Theorem

Let A be an $n \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, and let P be the parallelepiped determined by A. Then

$$(\text{volume of } P) = |\det(A)|.$$

Sanity check: the volume of P is zero $\iff$ the columns are *linearly dependent* (P is "flat") $\iff$ the matrix A is not invertible.

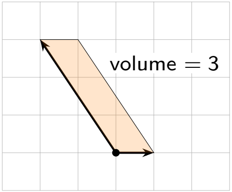
Why is the theorem true? You only have to check that the volume behaves the same way under row operations as $|\det|$ does.

Note that the volume of the unit cube (the parallelepiped defined by the identity matrix) is 1.

Determinants and Volumes

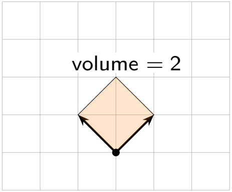
Examples in $\mathbf{R}^2$

$$\det\begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} = 3$$

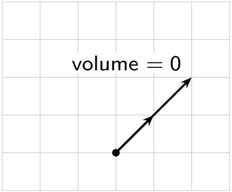


$$\det\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = -2$$

(Should the volume really be -2?)



$$\det\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = 0$$



Determinants and Volumes

Theorem

Let A be an $n \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, and let P be the parallelepiped determined by A. Then

$$(\text{volume of } P) = |\det(A)|.$$

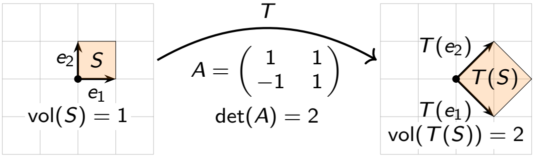
This is even true for curvy shapes, in the following sense.

Theorem

Let A be an $n \times n$ matrix, and let $T(x) = Ax$. If S is any region in $\mathbf{R}^n$, then

$$(\text{volume of } T(S)) = |\det(A)|\,(\text{volume of } S).$$

If S is the unit cube, then $T(S)$ is the parallelepiped defined by the columns of A, since the columns of A are $T(e_1), T(e_2), \ldots, T(e_n)$. In this case, the second theorem is the same as the first.

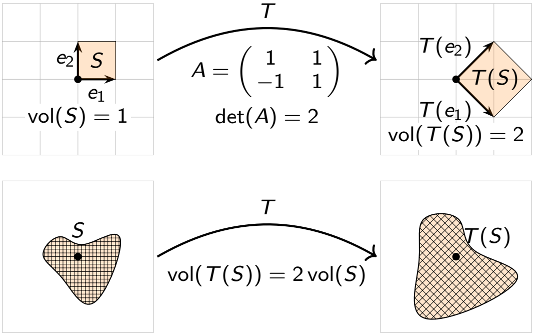


Theorem

Let A be an $n \times n$ matrix, and let $T(x) = Ax$. If S is any region in $\mathbf{R}^n$, then

$$(\text{volume of } T(S)) = |\det(A)|\,(\text{volume of } S).$$

For curvy shapes, you break S up into a bunch of tiny cubes. Each one is scaled by $|\det(A)|$; then you use *calculus* to reduce to the previous situation!



Theorem

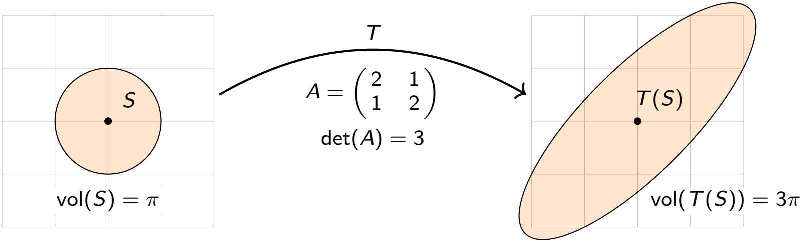
Let A be an $n \times n$ matrix, and let $T(x) = Ax$. If S is any region in $\mathbf{R}^n$, then

$$(\text{volume of } T(S)) = |\det(A)|\,(\text{volume of } S).$$

Example: Let S be the unit disk in $\mathbf{R}^2$, and let $T(x) = Ax$ for

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Note that $\det(A) = 3$.



Summary

Magical Properties of the Determinant

you really have to know these

1. There is one and only one function $\det\colon \{\text{square matrices}\} \to \mathbf{R}$ satisfying the properties (1)–(4) on the second slide.
2. A is invertible if and only if $\det(A) \neq 0$.
3. The determinant of an upper- or lower-triangular matrix is the product of the diagonal entries.
4. If we row reduce A to row echelon form B using r swaps, then
$$\det(A) = (-1)^r \frac{(\text{product of the diagonal entries of } B)}{(\text{product of the scaling factors})}.$$
5. $\det(AB) = \det(A)\det(B)$ and $\det(A^{-1}) = \det(A)^{-1}$.
6. $\det(A) = \det(A^T)$.
7. $|\det(A)|$ is the volume of the parallelepiped defined by the columns of A.
8. If A is an $n \times n$ matrix with transformation $T(x) = Ax$, and S is a subset of $\mathbf{R}^n$, then the volume of $T(S)$ is $|\det(A)|$ times the volume of S. (Even for curvy shapes S.)