Section 3.2

One-to-one and Onto Transformations

Matrix Transformations

Reminder

Recall: Let A be an $m \times n$ matrix. The **matrix transformation** associated to A is the transformation

$$T\colon \mathbf{R}^n \longrightarrow \mathbf{R}^m \qquad \text{defined by} \qquad T(x) = Ax.$$

- The *domain* of T is $\mathbf{R}^n$, which is the number of *columns* of A.
- The *codomain* of T is $\mathbf{R}^m$, which is the number of *rows* of A.
- The *range* of T is the set of all images of T:

$$T(x) = Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

This is the *column space* of A. It is a span of vectors in the codomain.

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T \colon \mathbf{R}^2 \to \mathbf{R}^3$.

- If $u = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ then $T(u) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 7 \end{pmatrix}$.
- Let $b = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$. Find v in $\mathbf{R}^2$ such that $T(v) = b$. Is there more than one?

  We want to find v such that $T(v) = Av = b$. We know how to do that:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} v = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix} \xrightarrow{\text{augmented matrix}} \left(\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 5 \\ 1 & 1 & 7 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right).$$

  This gives $x = 2$ and $y = 5$, or $v = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ (unique). In other words,

$$T(v) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}.$$

Matrix Transformations

Example, continued

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathbf{R}^2 \to \mathbf{R}^3$.

- Is there any c in $\mathbf{R}^3$ such that there is more than one v in $\mathbf{R}^2$ with $T(v) = c$?

  Translation: is there any c in $\mathbf{R}^3$ such that the solution set of $Ax = c$ has more than one vector v in it?

  The solution set of $Ax = c$ is a translate of the solution set of $Ax = b$ (from before), which has one vector in it. So the solution set to $Ax = c$ has only one vector. So no!

- Find c such that there is *no* v with $T(v) = c$.

  Translation: Find c such that $Ax = c$ is inconsistent.

  Translation: Find c not in the column space of A (i.e., the range of T).
  We could draw a picture, or notice that if $c = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, then our matrix equation translates into

  $$x + y = 1 \qquad y = 2 \qquad x + y = 3,$$

  which is obviously inconsistent.

Note: All of these questions are questions about *the transformation* T; it still makes sense to ask them in the absence of the matrix A.

The fact that T comes from a matrix means that these questions translate into questions about a matrix, which we know how to do.

Non-example: $T\colon \mathbf{R}^2 \to \mathbf{R}^3 \qquad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin x \\ xy \\ \cos y \end{pmatrix}$

Question: Is there any c in $\mathbf{R}^3$ such that there is more than one v in $\mathbf{R}^2$ with $T(v) = c$?

Note the question still makes sense, although T has no hope of being a matrix transformation.

By the way,

$$T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin 0 \\ 0 \cdot 0 \\ \cos 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin \pi \\ 0 \cdot \pi \\ \cos 0 \end{pmatrix} = T\begin{pmatrix} \pi \\ 0 \end{pmatrix},$$

so the answer is yes.

Questions About Transformations

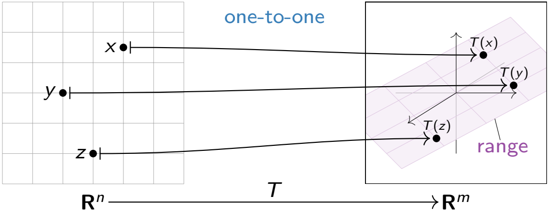
Today we will focus on two important questions one can ask about a transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$:

- Do there exist distinct vectors x, y in $\mathbf{R}^n$ such that $T(x) = T(y)$?
- For every vector v in $\mathbf{R}^m$, does there exist a vector x in $\mathbf{R}^n$ such that $T(x) = v$?

These are subtle because of the multiple *quantifiers* involved ("for every", "there exists").

Definition

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is **one-to-one** (or **into**, or **injective**) if different vectors in $\mathbf{R}^n$ map to different vectors in $\mathbf{R}^m$. In other words, for every b in $\mathbf{R}^m$, the equation $T(x) = b$ has *at most one* solution x. Or, different inputs have different outputs. Note that *not* one-to-one means at least two different vectors in $\mathbf{R}^n$ have the same image.



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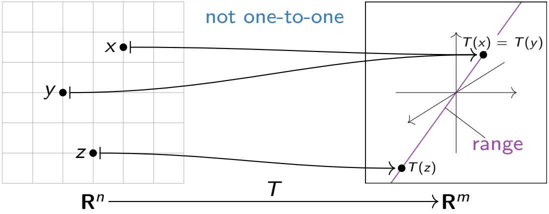

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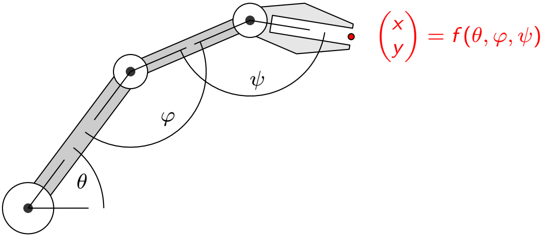
Poll

Consider the robot hand transformation from last lecture:



Define $f\colon \mathbf{R}^3 \to \mathbf{R}^2$ by:

$$f(\theta, \varphi, \psi) = \text{position of the hand at joint angles } \theta, \varphi, \psi.$$

Poll

Is f one-to-one?

No: there is more than one way to move the hand to the same point.

Characterization of One-to-One Matrix Transformations

Theorem

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a matrix transformation with matrix A. Then the following are equivalent:

- T is one-to-one.
- For each b in $\mathbf{R}^m$, the equation $T(x) = b$ has at most one solution.
- For each b in $\mathbf{R}^m$, the equation $Ax = b$ has a unique solution or is inconsistent.
- $Ax = 0$ has a unique solution.
- The columns of A are linearly independent.
- A has a pivot in every column.

Question

If $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is one-to-one, what can we say about the relative sizes of n and m?

Answer: T corresponds to an $m \times n$ matrix A. In order for A to have a pivot in every column, it must have *at least as many* rows as columns: $n \leq m$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

For instance, $\mathbf{R}^3$ is "too big" to map *into* $\mathbf{R}^2$.

Define

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad T(x) = Ax,$$

so $T\colon \mathbf{R}^2 \to \mathbf{R}^3$. Is T one-to-one?

The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

which has a pivot in every column. Hence T is one-to-one.

[interactive]

One-to-One Transformations

Non-Example

Define

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad T(x) = Ax,$$

so $T\colon \mathbf{R}^3 \to \mathbf{R}^2$. Is T one-to-one? If not, find two different vectors x, y such that $T(x) = T(y)$.

The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

which does not have a pivot in every column. Hence A is not one-to-one. In particular, $Ax = 0$ has nontrivial solutions. The parametric form of the solutions of $Ax = 0$ are

$$\begin{aligned} x \qquad - z &= 0 \\ y + z &= 0 \end{aligned} \implies \begin{aligned} x &= z \\ y &= -z. \end{aligned}$$

Taking $z = 1$ gives

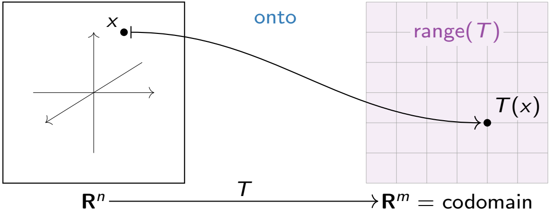
$$T\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0 = T\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

[interactive]

Onto Transformations

Definition

A transformation $T: \mathbf{R}^n \to \mathbf{R}^m$ is **onto** (or **surjective**) if the range of T is equal to $\mathbf{R}^m$ (its codomain). In other words, for every b in $\mathbf{R}^m$, the equation $T(x) = b$ has at *least one solution*. Or, every possible output has an input. Note that *not* onto means there is some b in $\mathbf{R}^m$ which is not the image of any x in $\mathbf{R}^n$.



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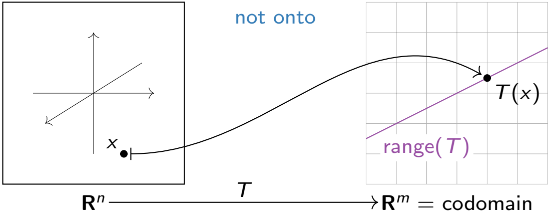

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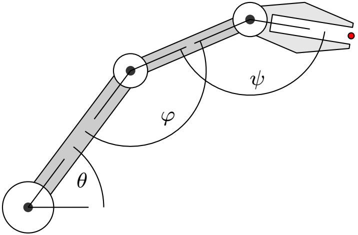
Poll

Consider the robot hand transformation again:



$$\begin{pmatrix} x \\ y \end{pmatrix} = f(\theta, \varphi, \psi)$$

Define $f\colon \mathbf{R}^3 \to \mathbf{R}^2$ by:

$$f(\theta, \varphi, \psi) = \text{position of the hand at joint angles } \theta, \varphi, \psi.$$

Poll

Is f onto?

No: it can't reach points that are far away.

Characterization of Onto Matrix Transformations

Theorem

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a matrix transformation with matrix A. Then the following are equivalent:

- T is onto
- $T(x) = b$ has a solution for every b in $\mathbf{R}^m$
- $Ax = b$ is consistent for every b in $\mathbf{R}^m$
- The columns of A span $\mathbf{R}^m$
- A has a pivot in every row

Question

If $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is onto, what can we say about the relative sizes of n and m?

Answer: T corresponds to an $m \times n$ matrix A. In order for A to have a pivot in every row, it must have *at least as many* columns as rows: $m \le n$.

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \end{pmatrix}$$

For instance, $\mathbf{R}^2$ is "too small" to map *onto* $\mathbf{R}^3$.

Define

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad T(x) = Ax,$$

so $T\colon \mathbf{R}^3 \to \mathbf{R}^2$. Is T onto?

The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

which has a pivot in every row. Hence T is onto.

Note that T is *onto* but not *one-to-one*.

[interactive]

Define

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad T(x) = Ax,$$

so $T\colon \mathbf{R}^2 \to \mathbf{R}^3$. Is T onto? If not, find a vector v in $\mathbf{R}^3$ such that there does not exist any x in $\mathbf{R}^2$ with $T(x) = v$.

The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

which does not have a pivot in every row. Hence A is not onto.
In order to find a vector v not in the range, we notice that $T\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = \left(\begin{smallmatrix} a \\ b \\ a \end{smallmatrix}\right)$. In particular, the x- and z-coordinates are the same for every vector in the range, so for example, $v = \left(\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right)$ is not in the range.

Note that T is *one-to-one* but not *onto*.

[interactive]

Define

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix} \qquad T(x) = Ax,$$

so $T\colon \mathbf{R}^3 \to \mathbf{R}^2$. Is T one-to-one? Is it onto?

The reduced row echelon form of A is

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix},$$

which does not have a pivot in every row or in every column. Hence T is neither one-to-one nor onto.

[interactive]

Summary

- A transformation T is **one-to-one** if $T(x) = b$ has *at most one* solution, for every b in $\mathbf{R}^m$.
- A transformation T is **onto** if $T(x) = b$ has *at least one* solution, for every b in $\mathbf{R}^m$.
- A matrix transformation with matrix A is one-to-one if and only if the columns of A are linearly independent, if and only if A has a pivot in every column.
- A matrix transformation with matrix A is onto if and only if the columns of A span $\mathbf{R}^m$, if and only if A has a pivot in every row.
- Two of the most basic questions one can ask about a transformation is whether it is one-to-one or onto.